Basic Mathematics for Understanding Noisy Waveforms and Frequency Spectra

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Notes prepared for Yale NPA/WIDG seminar

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This work combines three separate disciplines
It is a bit heavy and requires several sittings.

1) ***Understanding and modeling physical systems such as electronics or mechanical systems.***
2) ***Elementary Mathematical analysis and Fourier transforms.***
3) ***Some elements of probability and statistics.***

None of this is original, but I have assembled it in a way that should be useful for detector physicists. Some derivations are unusual to make them much more relatable.

Outline

- ***In part 1***
- ***Some examples of linear systems.***
- ***Introduction to Fourier transforms and discrete Fourier transform.***
- ***Definitions regarding noise waveforms, and its Gaussian character.***
- ***Part 2***
- ***White noise***
- ***Gaussian noise and central limit theorem.***
- ***Relation to Brownian motion.***
- ***Classification of noise spectra.***

White noise

Take time interval $0 \rightarrow T$

Assume there are N elementary noise excitations in this time interval.

Each characterized by q_i

$$e(t) = \sum_{i=1}^{N} q_i \delta(t - t_i)$$

If each elementary excitation produces a response $g(t)$ (with Fourier transform G(ω))

then the noise waveform is given by

$$eg(t) = \sum_{i=1}^{N} q_i g(t - t_i)$$

Fourier transform of this is given by

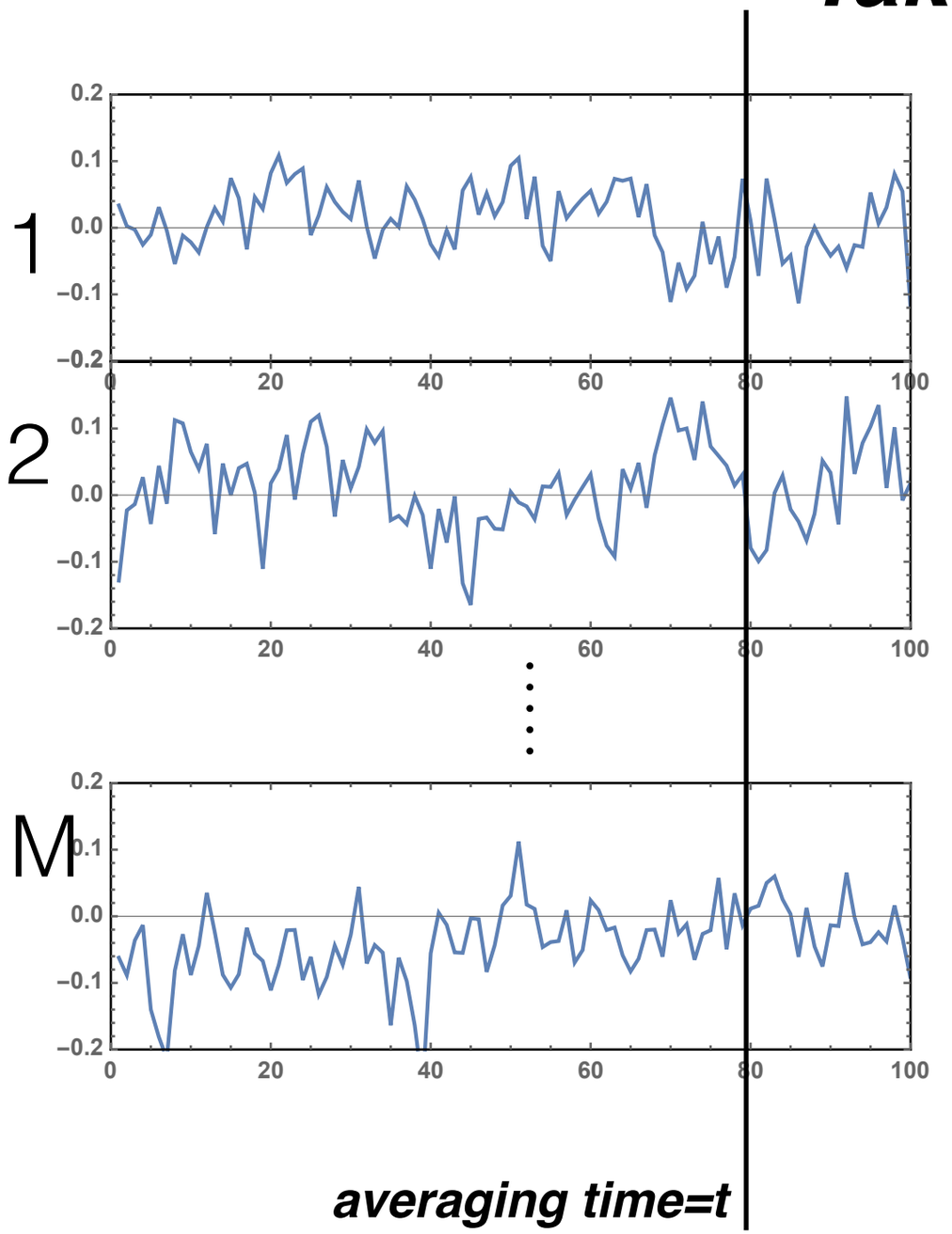
$$EG(\omega) = \sum_{i=1}^{N} q_i G(\omega) e^{-i\omega t_i}$$

We have to examine this function as rate $r = N / T \rightarrow \infty$

Question: What is the mean value of eg(t) and the variance of eg(t) ?

and what is the probability density function ?

Taking the average



The average value of $eg(t)$ is not an average over time ($0 \rightarrow T$), but an average over great many intervals of length T with t held fixed. Imagine that g(t) is a pulse much shorter than T, and rate r =N/T constant.

Then let the number of intervals go to infinity.

$$\langle eg(t)\rangle = r\langle q\rangle \int_{-\infty}^{\infty} g(t)\,dt$$

This is the average value of the pulse multiplied by the rate.

$$\langle eg(t)^2\rangle - \langle eg(t)\rangle^2 = r\langle q^2\rangle \int_{-\infty}^{\infty} g(t)^2\,dt$$

This proof takes a bit of work, ...

This is called Campbell's theorem. In its most general form, the averages of powers of g(t) form the coefficients of expansion of the logarithm of the charateristic function. We leave this for you to investigate.

Autocorrelation and the power spectrum

$ac(\tau) = \int_{-\infty}^{\infty} eg(t) \cdot eg(t+\tau) dt$ is the autocorrelation function for the noise.

Convert to Fourier transform $EG(\omega) = Fourier[eg(t)]$

$$ac(\tau) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \, EG(\omega') e^{i\omega' t} EG(\omega) e^{i\omega t} e^{i\omega\tau}$$

First integrate over t

$$ac(\tau) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega (2\pi\delta(\omega+\omega')) EG(\omega) EG(\omega') e^{i\omega\tau}$$

Switch $\omega' \rightarrow -\omega$, recall that EG is Hermitian $EG(-\omega) = EG^*(\omega)$

$$ac(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \left| EG(\omega) \right|^2 e^{i\omega\tau}$$

This is called the Weiner-Khinchin theorem. It states that the autocorrelation is the inverse Fourier transform of the power spectrum of the noise.

Therefore only the power spectrum is needed to specify the characteristics of the noise. When τ=0, we end up with Parseval's theorem.

$\int_{-\infty}^{\infty} eg(t)^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \left| EG(\omega) \right|^2$ We now need to figure this in terms of g(t) or G(ω)

This is the simple version of this

We now work on the previous result for white noise and relate it to the impulse response function of the system.

$$ac(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \left|EG(\omega)\right|^2 e^{i\omega\tau}$$

$$ac(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \sum_{i=1}^{N}\sum_{j=1}^{N} q_i q_j G(\omega)G^*(\omega)e^{-i\omega(t_i - t_j)}e^{i\omega\tau}$$

as N $\rightarrow \infty$ and if all q_i have the same magnitude

$$ac(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega G(\omega)G^*(\omega) \bullet N \bullet \left|q\right|^2 \bullet e^{i\omega\tau}$$

If we set $q = \pm Q/\sqrt{N}$ where Q is the elementary noise charge then

$$ac(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \left|G(\omega)\right|^2 \bullet Q^2 \bullet e^{i\omega\tau}$$

This shows that the noise characteristics for white noise after a filter are only dependent on the power spectrum of the filter. I have played a small trick to make this simple.

And so we finally have

$$\left\langle eg(t)^2 \right\rangle - \left\langle eg(t) \right\rangle^2 = r \times Q^2 \int_{-\infty}^{\infty} g(t)^2 \, dt = r \times \frac{Q^2}{2\pi} \int_{-\infty}^{\infty} \left| G(\omega) \right|^2 d\omega$$

This states that variance (RMS) of the noise depends on the filter power spectrum. The argument was done starting with white noise, but is general. The noise has a finite variance only if the integral converges. This depends on the nature of the filter function and its poles.

What do we do if the noise at the input is not white ?

a) Numerically, we can absorb the additional frequency dependence into $G(\omega)$

b) Clearly the native input noise must also be limited in frequency domain and so additional terms may be needed in the mathematical model.

c) The best might be to use an empirical model using measurements of the system.

We will examine this later ....

Closer look at white noise

White noise has constant power at all frequencies since Fourier transform of delta functions is a constant. This means equal amount of fluctuation no matter how finely the time is divided or an infinite fluctuation at a single moment.

$$e(t) = \sum_{i=1}^{N} q_i \delta(t - t_i) \Rightarrow \text{Fourier transform } \Rightarrow E(\omega) = \sum_{i=1}^{N} q_i e^{-i\omega t_i}$$

$$|E(\omega)|^2 = \sum_i \sum_j e^{-i\omega(t_i - t_j)} q_i q_j \quad \text{and } ac(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega |E(\omega)|^2 e^{i\omega\tau}$$

Take the average of the sum over all frequencies in the integral.

$$ac(\tau) = \frac{1}{2\pi} \int_{-W}^{W} \sum_i \sum_j e^{-i\omega(t_i - t_j)} q_i q_j e^{i\omega\tau} \, d\omega \ldots \text{ we let N, W} \to \infty$$

oscillating terms in the sums will average to zero unless $t_i = t_j$ giving just the total N.

$$ac(\tau) = \int_{-\infty}^{\infty} d\omega \frac{Nq^2}{2\pi} e^{i\omega\tau} \Rightarrow \text{ Flat noise power. ac(0) (total power) is undefined.}$$

The total integrated power in white noise is undefined because there is fluctuation at all time scales, no matter how finely you divide the time (or how high the frequency).

At this point it is useful to learn the central limit theorem

Let X_i be N independent random variables with probability densities $P_{X_i}(x_i)$

Recall that the characteristic function is $\varphi_{X_i}(k_i) = \int e^{ikx_i} P_{X_i}(x_i)\,dx_i$

$z = f(x_i)$ is a real function of the random variables X_i

How do we calculate the PDF of Z? It is not simple for any f, but there are some simple relations

The expectation value for the mean and square (if they exist).

$$\langle Z \rangle = \int f(x_1, x_2, \ldots, x_N) P_{X_1}(x_1)\,dx_1 P_{X_2}(x_2)\,dx_2 \ldots P_{X_N}(x_N)\,dx_N$$

$$\langle Z^2 \rangle = \int (f(x_1, x_2, \ldots, x_N))^2 P_{X_1}(x_1)\,dx_1 P_{X_2}(x_2)\,dx_2 \ldots P_{X_N}(x_N)\,dx_N$$

Recall that the moments are defined in terms of derivatives of the characteristic function

Therefore the characteristic function for Z is

$$\varphi_Z(k) = \int e^{ikf(x_1, x_2, \ldots, x_N)} P_{X_1}(x_1)\,dx_1 P_{X_2}(x_2)\,dx_2 \ldots P_{X_N}(x_N)\,dx_N$$

It may not be simple to invert this for any function f to obtain the PDF for Z.

We will now do this for a simple case.

Let all X_i have the same PDF and $z = f(x_i) = \frac{1}{N}\sum x_i$;

in other words, the random variable Z is the average of random variables X_i

also assume that the expectation $\langle X \rangle = 0$ for simplicity. And let N be very large.

$$\varphi_Z(k) = \int e^{i\frac{k}{N}\sum x_i} P_{X_1}(x_1)dx_1 \ldots P_{X_N}(x_N)dx_N$$

$$= (\varphi_X(\frac{k}{N}))^N = \left(1 + \cancel{\langle x \rangle \frac{k}{N}} - \langle x^2 \rangle \frac{k^2}{2N^2} + O(k^4)..\right)^N$$

Recall $e = \lim_{N\to\infty}(1+1/N)^N$ and so finally. (with $\sigma^2 = \langle x^2 \rangle$)

$$\varphi_Z(k) \approx e^{-k^2\frac{\sigma^2}{2N}}$$

This is the characteristic function for a Gaussian distribution with variance of $\frac{\sigma^2}{N}$.

Recall from the earlier part of this lecture that the Gaussian PDF that corresponds to this characteristic function is

$$P_Z(z) = \frac{\sqrt{N}}{\sigma\sqrt{2\pi}} e^{-\frac{N \bullet z^2}{2\sigma^2}}$$

The result is much more general than indicated in this calculation. The sum of a large number of independent random variables approaches the Gaussian distribution. This is true even if the individual variables themselves are not Gaussian distributed.

Notice the conditions: $\langle x^2 \rangle$ must exist ! And the Gaussian limit is obtained as N $\to \infty$

white noise distribution after digitization

Let start with white noise over period T.

$$e(t)=\sum_{i=1}^{N} q_i\delta(t-t_i) \Rightarrow \text{Fourier transform } \Rightarrow E(\omega)=\sum_{i=1}^{N} q_i e^{-i\omega t_i}$$

we will set $q_i = s_i \frac{Q}{\sqrt{N}}$ ..... $E(\omega)=Q\sum_i \frac{s_i e^{-i\omega t_i}}{\sqrt{N}}$ here $s_i = \pm 1$

In the case of a discrete spectrum how do we end up with Gaussian distributed values ?

Take time interval T with M bins of width $\Delta = \frac{T}{M}$ and $\omega_l = l\cdot\frac{2\pi}{T}$, place N pulses in this time.

$$F(\omega)=\frac{1}{\sqrt{N}}\sum_{i=1}^{N} s_i e^{-i\omega t_i} \Rightarrow F_l = \frac{1}{\Delta\sqrt{N}}\sum_{i=1}^{N} s_i e^{-i2\pi l R_i}$$ where R_i is uniform random 0-1

This is a 2 dim random walk with $F_l = x_l + iy_l$. (Take $F_0 = \frac{1}{\sqrt{N}}\sum s_i = 0$ for convenience.)

Now we take a inverse DFT of F_l (recall that sums over sinusoidals of diff. freq. add to 0)

$$f_k = \frac{1}{M}\sum_{l=-(M-1)/2}^{(M-1)/2} F_l\, e^{i2\pi l\frac{k}{M}} = \frac{1}{M}\sum_{l=1}^{(M-1)/2} 2x_l\cos(2\pi lk/M)+2y_l\sin(2\pi lk/M)$$

f_k is simply a Gaussian variate with $\sigma=1$. The expectation value $\langle f_k^2\rangle = 1$ **Prove this now.**

Notice that we effectively cutoff the high frequencies by digitizing the waveform in frequency space. And so the resulting white noise waveform does not have infinite power.

We examine this number somewhat more.

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e^{i2\pi t_i}\equiv X+iY=R\cdot e^{i\phi}$$ where t_i is a uniform random (0,1]

This is a very famous random variate called the 2D random walk.

(or 2D Brownian motion)

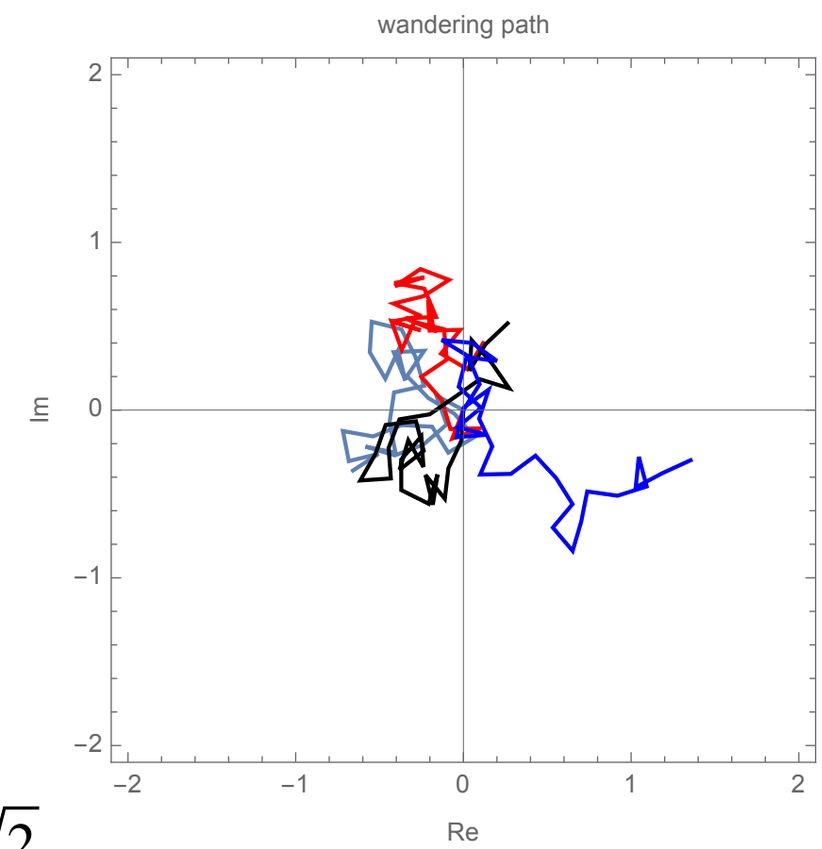
For large N the solution was given by Rayleigh (for step size of a) for the magnitude R.

$$P_N(R) = \frac{2R}{a^2N}e^{-\frac{R^2}{a^2N}}\to 2R\cdot e^{-R^2}$$ the ϕ is uniform 0-2π

By making the step size $a=\frac{1}{\sqrt{N}}$, we get a simple distribution.

The distribution for X and Y are simply normal Gaussian with $\sigma=1/\sqrt{2}$.

$$P(X)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{X^2}{2\sigma^2}}$$



Homework: can you imagine a simple way to generate a Gaussian variate ?

We will use the central limit theorem to prove this.

Historical interlude

The Problem of the Random Walk.

CAN any of your readers refer me to a work wherein I should find a solution of the following problem, or failing the knowledge of any existing solution provide me with an original one? I should be extremely grateful for aid in the matter.

A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r+\delta r$ from his starting point, O.

The problem is one of considerable interest, but I have only succeeded in obtaining an integrated solution for *two* stretches. I think, however, that a solution ought to be found, if only in the form of a series in powers of $1/n$, when n is large. KARL PEARSON.

The Gables, East Ilsley, Berks.

The Problem of the Random Walk.

THIS problem, proposed by Prof. Karl Pearson in the current number of NATURE, is the same as that of the composition of n iso-periodic vibrations of unit amplitude and of phases distributed at random, considered in *Phil. Mag.*, x., p. 73, 1880; xlvii., p. 246, 1899; ("Scientific Papers," i., p. 491, iv., p. 370). If n be very great, the probability sought is

$$\frac{2}{n}e^{-r^2/n}r\,dr.$$

Probably methods similar to those employed in the papers referred to would avail for the development of an approximate expression applicable when n is only moderately great. RAYLEIGH.

Terling Place, July 29.

Nature 72, 294; 318-342(1905)

The problem of the random walk arises in many situations.

Brownian motion - Einstein (1905)

Stock markets - Original formulation by Louis Bachelier (1900)

It is the basis of the advanced financial mathematics that is currently taught in business schools.

Now we apply the central limit theorem to the random walk function.

We assume that the step size is real $a > 0$

$$\sum_{i=1}^{N} a e^{i2\pi t_i} = a\sum_{i=1}^{N} \mathrm{Cos}(2\pi t_i) + i\,\mathrm{Sin}(2\pi t_i) \equiv X + iY$$

Here t_i are the random variables with support [0,1]. To find the distribution of X, and Y it is not necesssary to find the PDF for $\mathrm{Cos}(2\pi t_i)$ and $\mathrm{Sin}(2\pi t_i)$.

It is only necessary to find $\langle X \rangle$ and $\langle X^2 \rangle$ and calculate the Gaussian variance.

$$\langle X \rangle = a\sum_{i=1}^{N} \langle \mathrm{Cos}(2\pi t_i) \rangle = 0 = \langle Y \rangle$$

$$\langle X^2 \rangle = a^2 \sum_{i=1}^{N} \langle \mathrm{Cos}^2(2\pi t_i) \rangle = N\frac{a^2}{2} = \langle Y^2 \rangle$$

Both X, and Y are Gaussian distributed with variance of $\sigma^2 = \dfrac{Na^2}{2}$ by the CLT.

Notice that as $N \to \infty$, the mean remains zero, but because the variance blows up and the walker wanders away to infinity only as $\sigma \sim \sqrt{N}$ as we expected. To be explicit, the joint PDF

$$P(x)P(y) = \frac{1}{2\pi\left(Na^2/2\right)} e^{-x^2\frac{1}{Na^2}} e^{-y^2\frac{1}{Na^2}} = \frac{1}{2\pi\left(Na^2/2\right)} e^{-(x^2+y^2)\frac{1}{Na^2}}$$

$$P(r,\theta) = P(x,y)\cdot\frac{\partial(x,y)}{\partial(r,\theta)} = \frac{1}{2\pi\left(Na^2/2\right)} e^{-(r^2)\frac{1}{Na^2}} \cdot r \quad \ldots\ldots$$ Change of variables to polar coords

$$P(r) = \frac{2r}{Na^2} e^{-\frac{r^2}{Na^2}} \quad \ldots\ldots$$ after integrating over $\theta : [0 - 2\pi]$.....This is the derivation of the Rayleigh distribution.

an overview of previous stuff

The chain of logic here is as follows

*W*hite noise is formed from an infinite number of elementary excitations,
each one of infinitsimal scale. They are a Poisson process with the rate going to infinity.
When we digitize white noise over a finite time width, we cutoff high and low frequencies
and end up with equal power in each frequency band. The probability density of
the amplitude at each frequency is given by a random walk. This results in a Gaussian
distribution for the time domain waveform.

Above logic does not apply perfectly to the real world because then we could not
distinguish bird chirping and water droplets falling.
1) The noise only approaches Gaussian limit for infinite rate.
2) The assumption of stationarity does not hold perfectly. There can be many vibration
modes that damp at different rates leading to interesting effects.
*i.e.*Stuff does not sound the same forward and backwards.
3) Elementary noise is not white.

How do we think about noise with some arbitrary (but finite) noise power as a function of frequency ?

Start with white noise in time T with N elementary pulses at random times t_i

$e(t) = \sum_{i=1}^{N} q_i \delta(t - t_i)$ and take Fourier transform

$$E(\omega) = \sum_{i=1}^{N} q_i e^{-i\omega t_i}$$

Now apply the shaping filter in Fourier space. We can include non-white effects in this also.

$$EG(\omega) = \sum_{i=1}^{N} q_i e^{-i\omega t_i} G(\omega)$$

Now we discretize ω. $\omega_l = l \bullet \frac{2\pi}{T}$ where $l = -(M-1)/2,...,(M-1)/2$

$T = M \bullet \Delta$,..... I am assuming that M is odd here for symmetry.

Care is needed to change units when made discrete. $G_l = G(\omega_l) \cdot \frac{1}{\Delta}$ which has a phase...

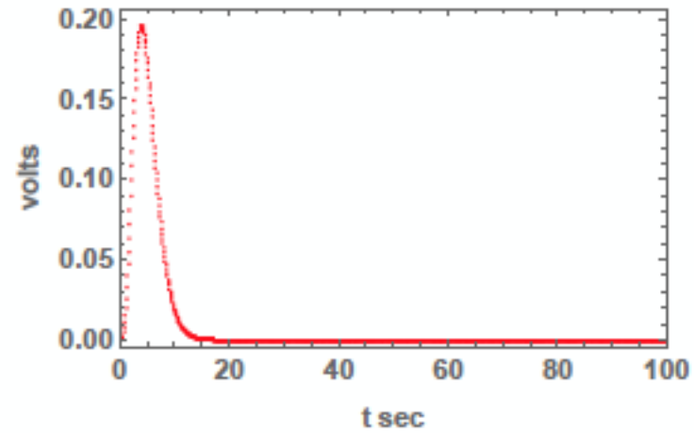
$$EG_l = \sum_{i=1}^{N} q_i \; e^{-i(2\pi/T)\cdot l \cdot Ran_i T} G(\omega_l) \cdot \frac{1}{\Delta} = \sum_{i=1}^{N} q_i \; e^{-i2\pi \cdot l \cdot Ran_i} G_l$$

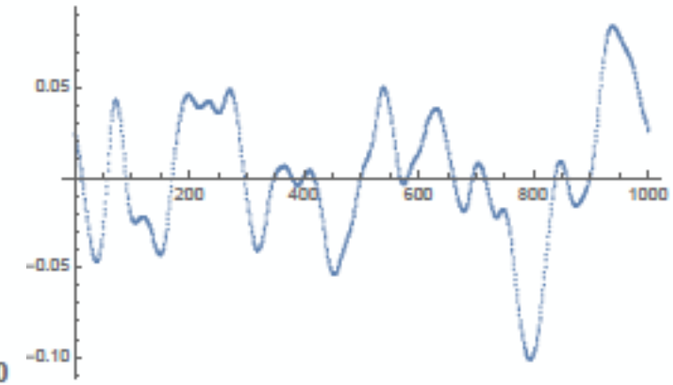
Ran_i is an array of random from 0 to 1. Must take care with the $2\pi l$

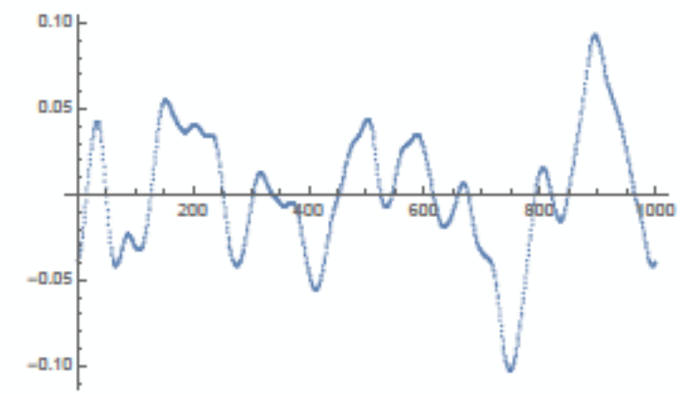
Notice $l = -(M-1)/2,... (M-1)/2$ making the formula manifestly $EG_{-l} = EG_l^*$

Therefore the resulting inverse transform from inverse DFT will be real.

Procedure for generating noise from a known spectral power







Choose the time interval. T in sec

Number of samples including 0. $M = 1001$

Sampling time Δ sec. Index $k = 0, \ldots M - 1$

Frequency index $l = -(M-1)/2, \ldots (M-1)/2$

Fundamental frequency $\delta\omega = \omega_0 = \dfrac{2\pi}{T}$

Generate two X, Y variates using Normal distribution with $\sigma = 1/\sqrt{2}$

From the known power spectrum select power for frequency $\omega = l \cdot \omega_0$

Set $G_l = |G(\omega_l)| / \Delta$ ... Check units.

Set simulated Fourier transform $EG_l = Q \cdot G_l \cdot (X + iY) = EG^*_{-l}$

Simulated noise waveform at each time k is given by where Q is the normalization.

$$NF_k = \frac{1}{\sqrt{M}} \sum_{l=-(M-1)/2}^{(M-1)/2} EG_l e^{i2\pi lk/M} \quad \text{where } k = -(M-1)/2, \ldots (M-1)/2$$

As shown above in the limit of large N (elementary pulses) the phase of G_l does not matter

It is possible to replace G_l with $Abs[G_l]$

This will symmetrize the noise pulse as the phase $\rightarrow 0$

*R*ecall that the inverse Fourier transform of a real function is symmetric around 0.

The spectral power remains the same even if the phase is dropped.

The general solution

The derivation provided above was deliberately made for digitized data to explicitly demostrate Gaussianity and also provide application method for many situations.

The more general solution is obtained from the following formula. The probability of the noise V (say it is a voltage) is given by.

$$P(V) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iVu} \, e^{r \int_{-\infty}^{\infty} (e^{iug(t)} - 1) dt} \, du \qquad \text{where } r = \text{rate}$$

Notice that this is an inverse of the characteristic function in red.
If the inner integrand is expanded then it is easy to see how one gets an Gaussian with variance of $\langle g(t)^2 \rangle - \langle g(t) \rangle^2$.

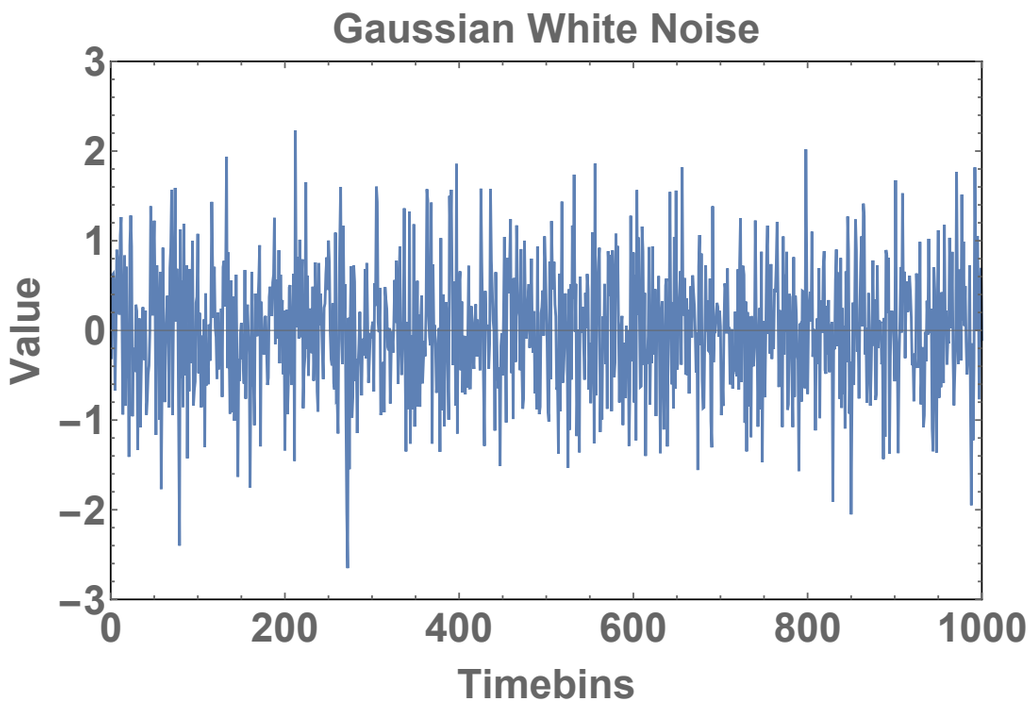
Rice provides a method for generalizing this to include the possibility that the noise pulses are distributed according to some arbitrary law (exponential for example).
Also look at "Algebra of Random Variables" by M.D. Springer.

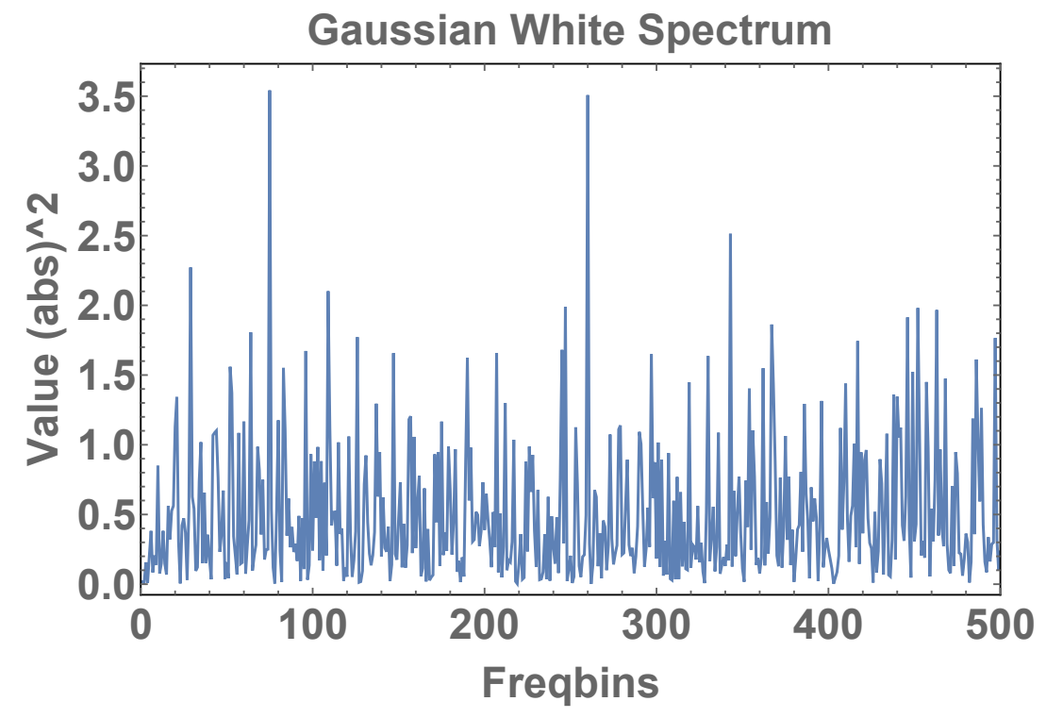
The issue of convergence.

There is a vast literature on this subject

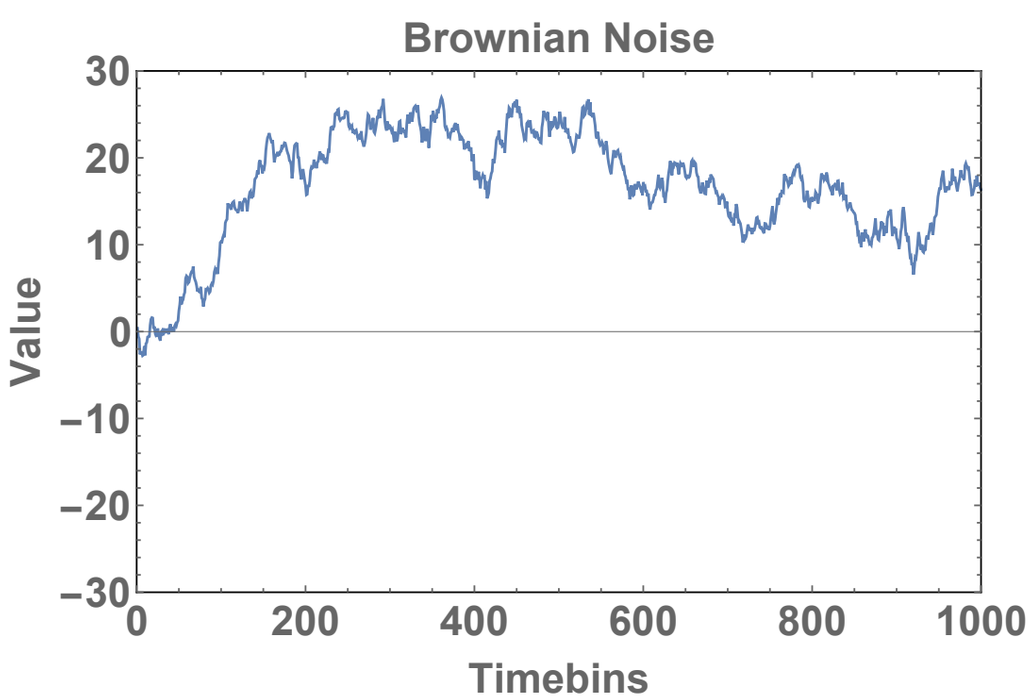
Noise type	Pulse Model	Power spectrum	Divergence	Practical devices
White	train of δ(t)	~ 1	Total noise diverges at infinite frequency	High frequency cutoff
Brownian motion	Integral of train of δ(t)	$\sim 1/f^2$	Total noise power diverges at 0 frequency	Low frequency cutoff
Flicker or 1/f	Train of pulses that have a $1/\sqrt{(t-t_0)}$ fall	$\sim 1/f$	Total noise diverge at both 0 and infinity	Both high and low cutoff

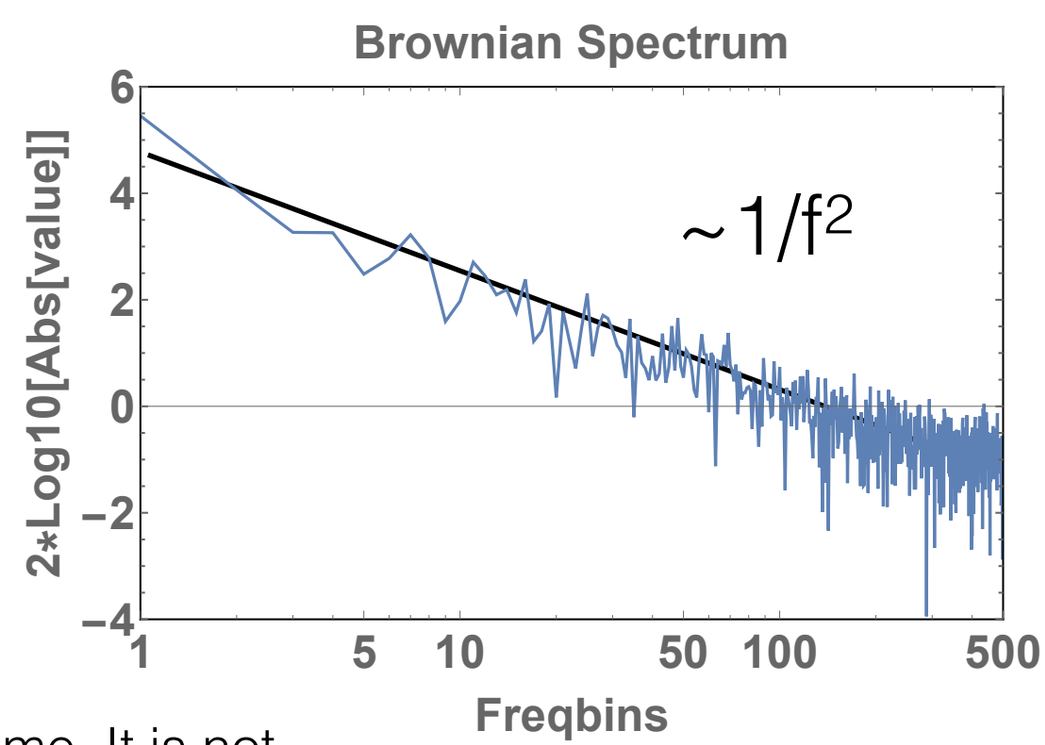
These infinities do not happen because practical devices (including analysis processing) have cutoffs at low and high frequencies. These cutoffs need to be understood to evaluate the result.





Integration of white noise is Brownian noise





eventually this looks the same forward or backwards in time. It is not possible to figure our the direction of time from a snippet of data.

closer look at Brownian motion noise

Part of the system noise could be due to integration of white noise.

What is the spectral power of such noise ?

Recall $Fourier[\int_{-\infty}^{t} f(x)dx] = \frac{1}{\sqrt{2\pi}}(\frac{F(\omega)}{i\omega} + \pi F(0)\delta(\omega))$

Start with white noise $\mathrm{e}(t) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} s_i \delta(t - t_i)$ $\quad s_i = \pm 1$ and t_i are random.

Fourier transform of the integral of $e(t)$ is

$$E(\omega) = \frac{1}{\sqrt{2\pi N}}\sum_{i=1}^{N} s_i(\frac{e^{-i\omega t_i}}{i\omega} + \pi\delta(\omega))$$ ...second term is the dc component

$$E(\omega)\cdot E^*(\omega) = \frac{1}{2\pi N}\left[\frac{N}{\omega^2} + \pi^2 \cancel{\sum s_i s_j \delta^2(\omega)}\right]$$ second term zero for no dc component

This converges at high ω, but diverges at zero.

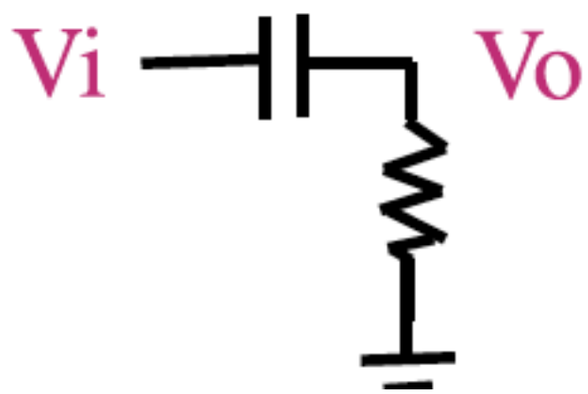
If there is an additional shaping filter $G(\omega)$ then the spectral power is given by

$$\mathrm{S}_{brownian}(\omega) = \frac{1}{2\pi}\left[\frac{G(\omega)\cdot G^*(\omega)}{\omega^2}\right]$$

To make sure this converges there must be a filter that cancels out the denominator.

Or in another words differentiates the waveform.

High pass, differentiator



With the application of a high pass filter with time constant τ the Brownian noise can be relaxed so that it does not diverge at low frequency.

$$S_{brownian-relaxed}(\omega)=\left[\frac{|G(\omega)|^2}{\omega^2}\times\frac{\omega^2}{(\omega^2+1/\tau^2)}\right]$$

$$S_{brownian-relaxed}(\omega)=\left[\frac{|G(\omega)|^2}{(\omega^2+1/\tau^2)}\right]$$

If G(ω)=1 then the filter simply is flat for ω<<1/τ and falls as $1/\omega^2$ for ω>>1/τ

$$V_o(t)+Q/C=V_i(t)$$

$$\frac{dV_o(t)}{dt}+I/C=\frac{dV_i(t)}{dt}$$

Use Fourier Transforms to solve

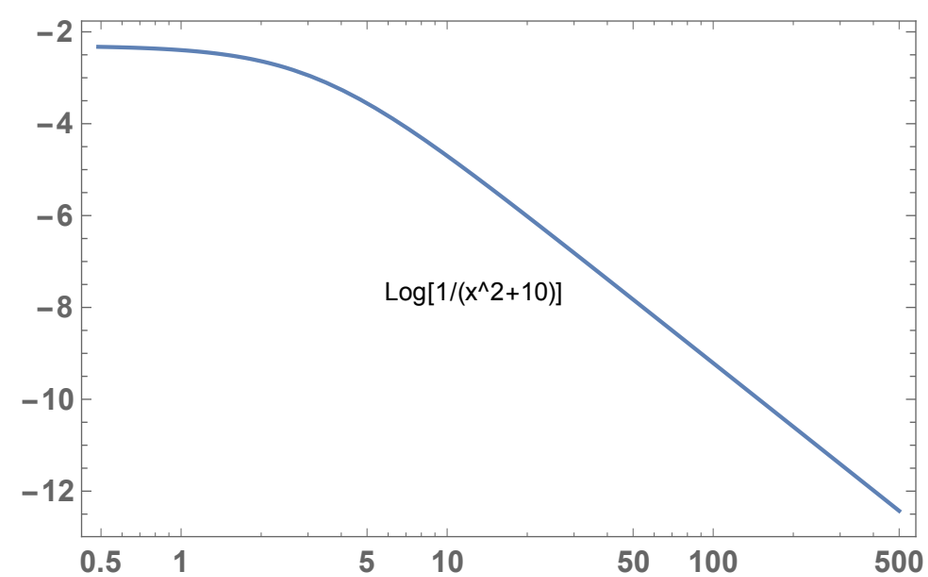
$$F\left[V(t)\right]=V(\omega),\ F[V'(t)]=i\omega V(\omega)$$

$$i\omega V_o(\omega)+V_o(\omega)/RC=i\omega V_i(\omega)$$

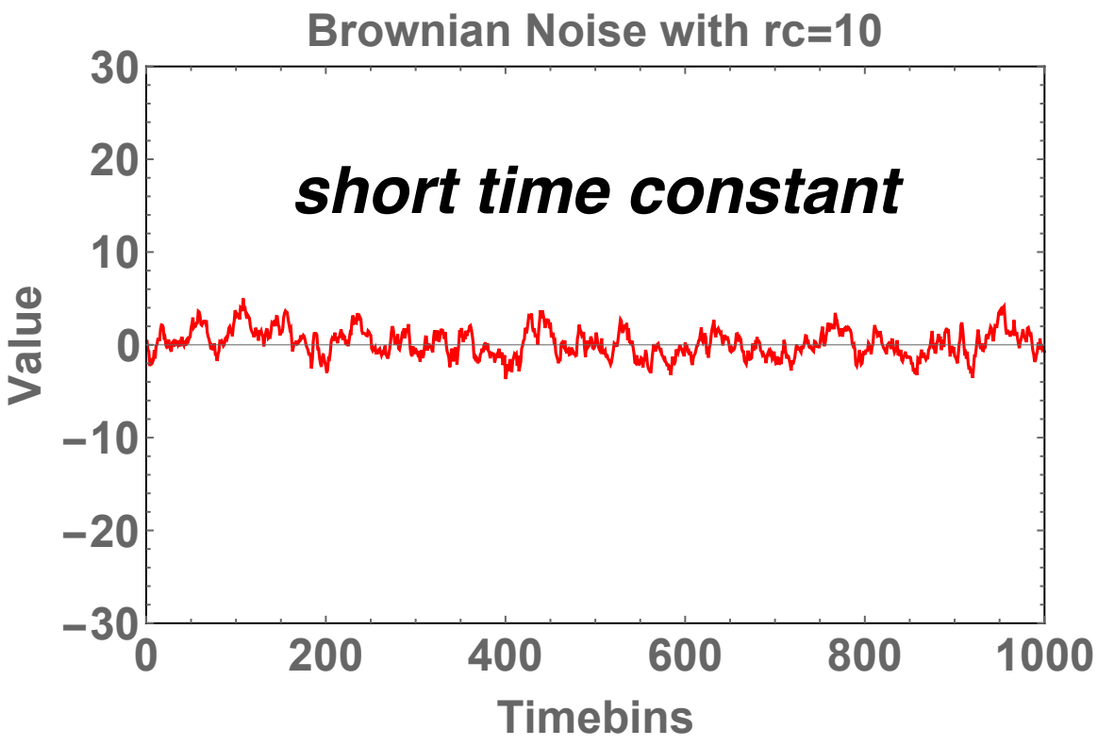
$$V_o(\omega)=V_i(\omega)\times\frac{i\omega}{i\omega+1/\tau}$$

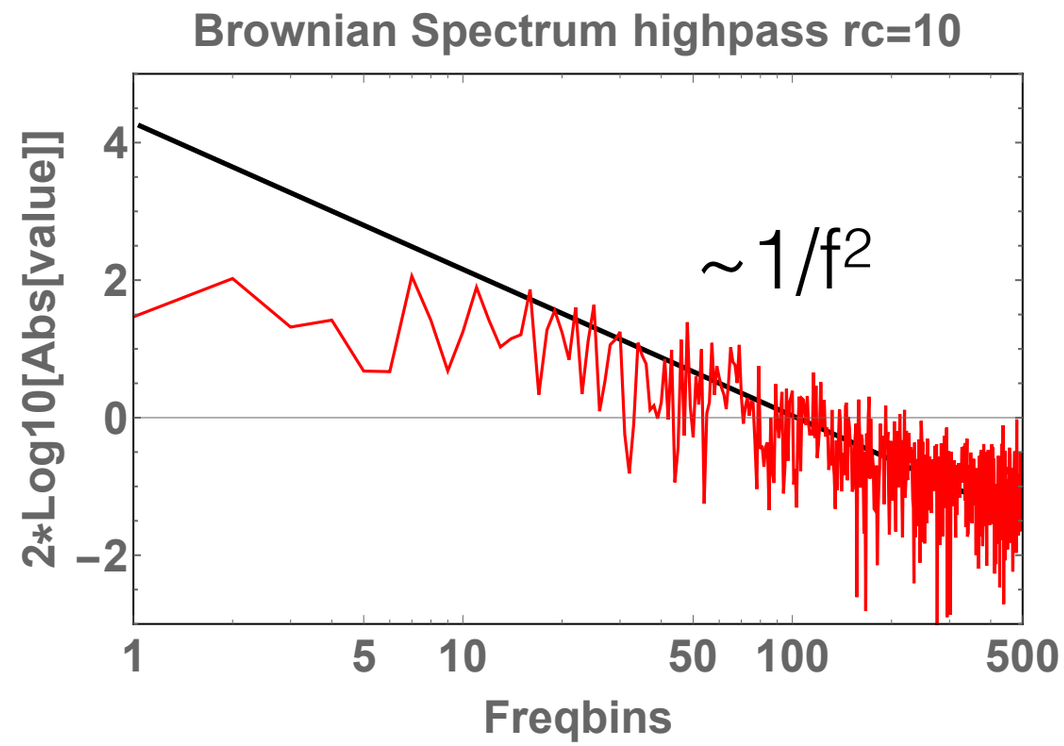
$$\left|\frac{V_o}{V_i}\right|^2=\frac{\omega^2}{(\omega^2+1/\tau^2)}$$

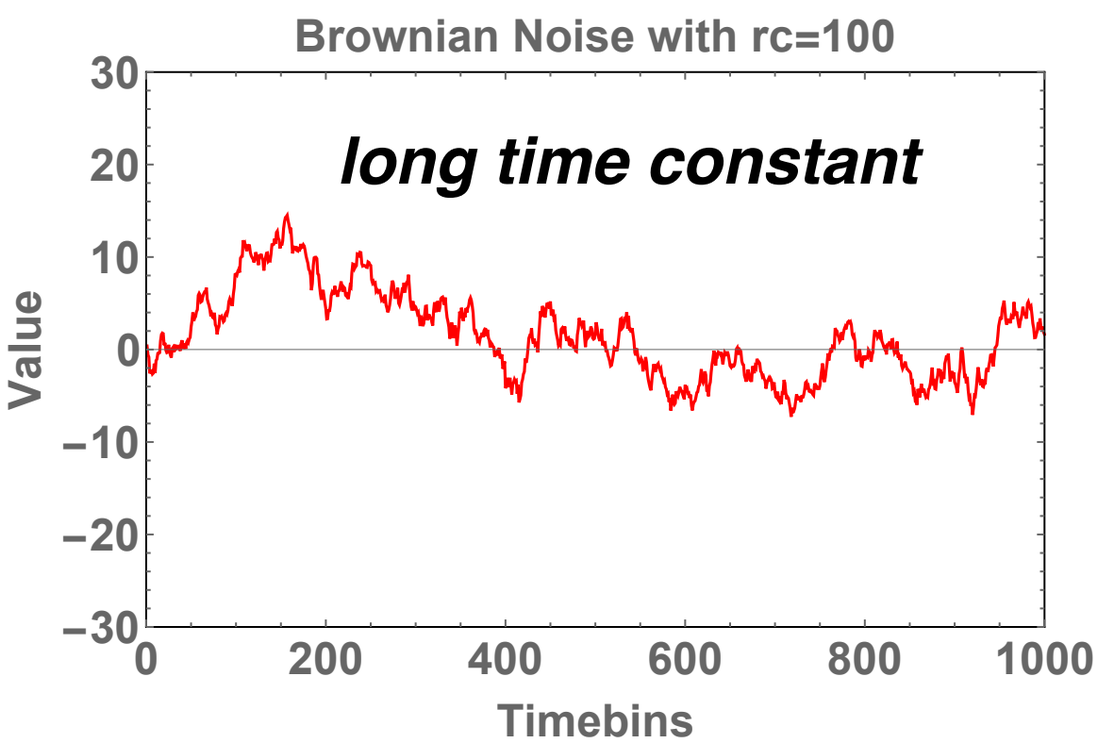
Large capacitors charge up spontaneously and need to have a discharging resistor even while being stored.

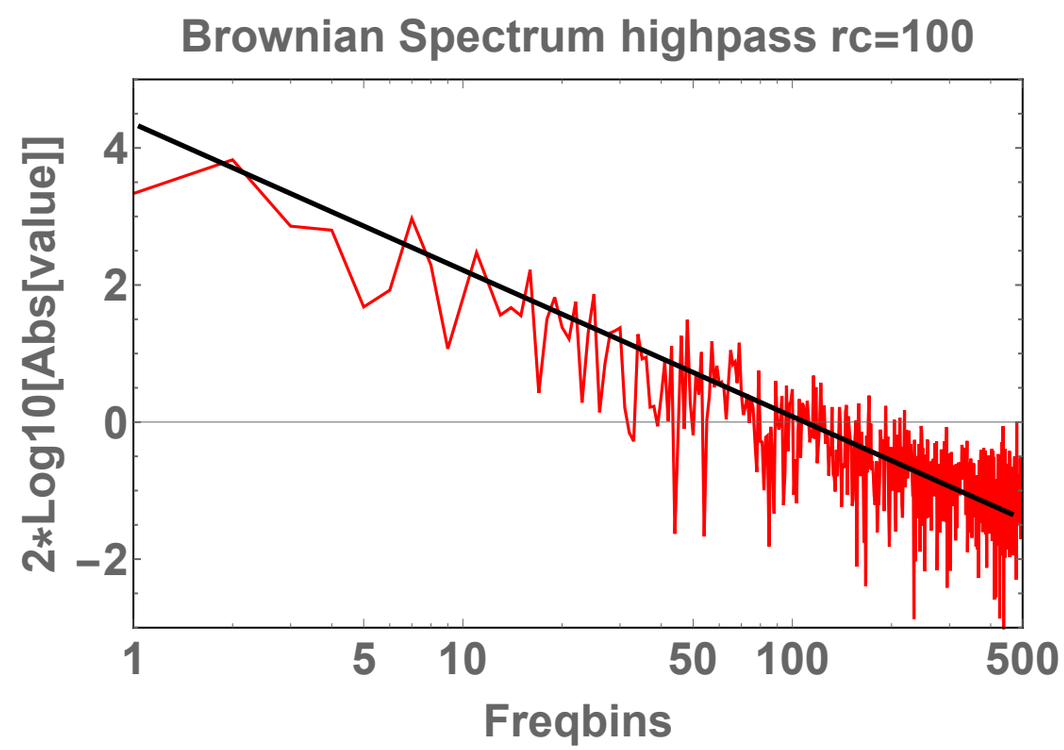


Brownian noise after application of high pass filter

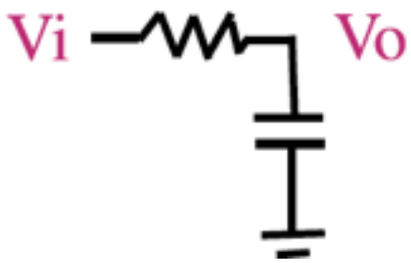


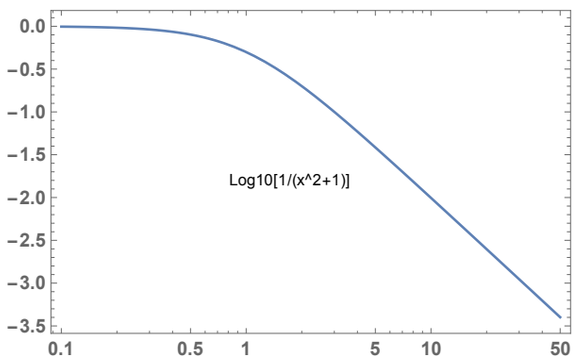






Origin of flicker noise



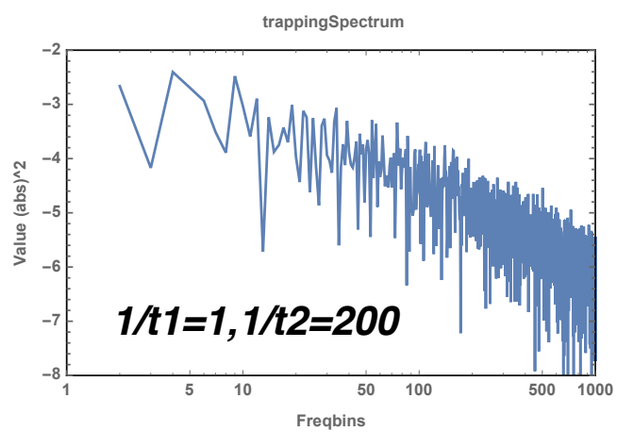


There does not seem to be an obvious mechanism that produces noise with spectrum that varies as $\sim 1/\omega$.

If the elementary pulses go through an exponential filter (τ=RC) then

$$EG(\omega)=\sum_{i=1}^{N} q_i e^{-i\omega t_i} G(\omega)=\sum_{i=1}^{N} q_i e^{-i\omega t_i}\times\frac{1}{1+i\omega\tau}$$

Power spectrum $S(\omega)=Q^2\dfrac{1}{1+(\omega\tau)^2}$ falls as $\sim 1/\omega^2$



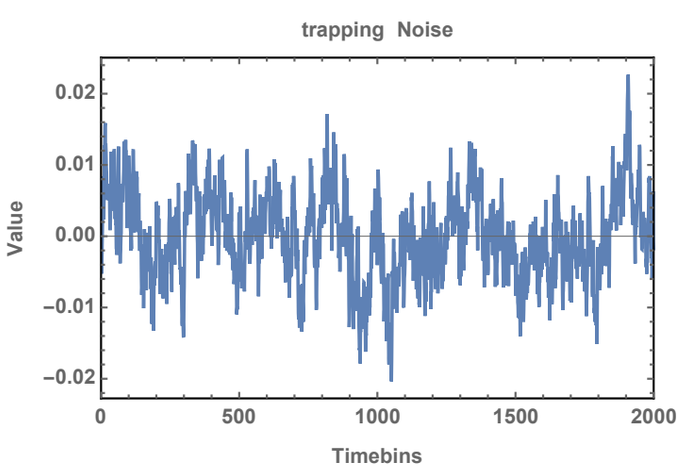
What if the time constant itself is uniform randomly changing between $1/\tau_1$ to $1/\tau_2$ then the spectrum has to be averaged over this

$$S(\omega)=\frac{1}{1/\tau_1-1/\tau_2}\int_{1/\tau_2}^{1/\tau_1} Q^2\frac{1}{1+(\omega/\lambda)^2}d\lambda$$

The solution to this has a region where

$$S(\omega)\approx\frac{\pi}{2(1/\tau_1-1/\tau_2)}\times\frac{1}{\omega}\quad\text{when } 1/\tau_2<\omega<1/\tau_1$$

We can do this by Monte Carlo.



Flicker noise. What kind of pulse in time domain ?

White noise has a flat frequency spectrum $S_w(\omega) \propto 1$

Brownian motion noise is the integral of white and $S_B(\omega) \propto \dfrac{1}{\omega^2}$

What kind of noise has a power law $S_F \propto \dfrac{1}{\omega^{1+\alpha}}$ with $\alpha < 1$?

Construct Fourier transform of $(1/\omega)$ noise: $F_F(\omega) = \dfrac{1}{\sqrt{N}} \sum_{i=1}^{N} e^{-i\omega t_i} \times \dfrac{1}{\sqrt{\omega}}$

(In mathematics this represents a fractional integral in time domain.)

Perform Inverse Fourier of $F_F(\omega)$. Just take one pulse at t_0

$$f_F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\omega}} e^{-i\omega t_0} e^{i\omega t} d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} \frac{1}{\sqrt{\omega}} e^{-i\omega t_0} e^{i\omega t} d\omega + \int_{0}^{\infty} \frac{1}{\sqrt{\omega}} e^{-i\omega t_0} e^{i\omega t} d\omega \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{\infty} \frac{1}{\sqrt{-\omega}} e^{+i\omega(t_0 - t)} d\omega + \int_{0}^{\infty} \frac{1}{\sqrt{\omega}} e^{+i\omega(t - t_0)} d\omega \right]$$

This integral has a branch cut because of the $\sqrt{\omega}$. The answer is

$$f_F(t) = \begin{cases} 2/\sqrt{t - t_0} & t > t_0 \quad = \quad J_+ \\ 0 & t \le t_0 \\ 0 & t \ge t_0 \\ 2/\sqrt{t_0 - t} & t < t_0 \quad = \quad J_- \end{cases}$$

A linear combination of J_+, J_-

This is a crazy integral

(advanced and retarded)
This is why this noise is not time reversible.
In retarded case, the noise dies off as $1/t^{0.5}$ after a pulse giving very long time correlations.

Examples of 1/f spectra

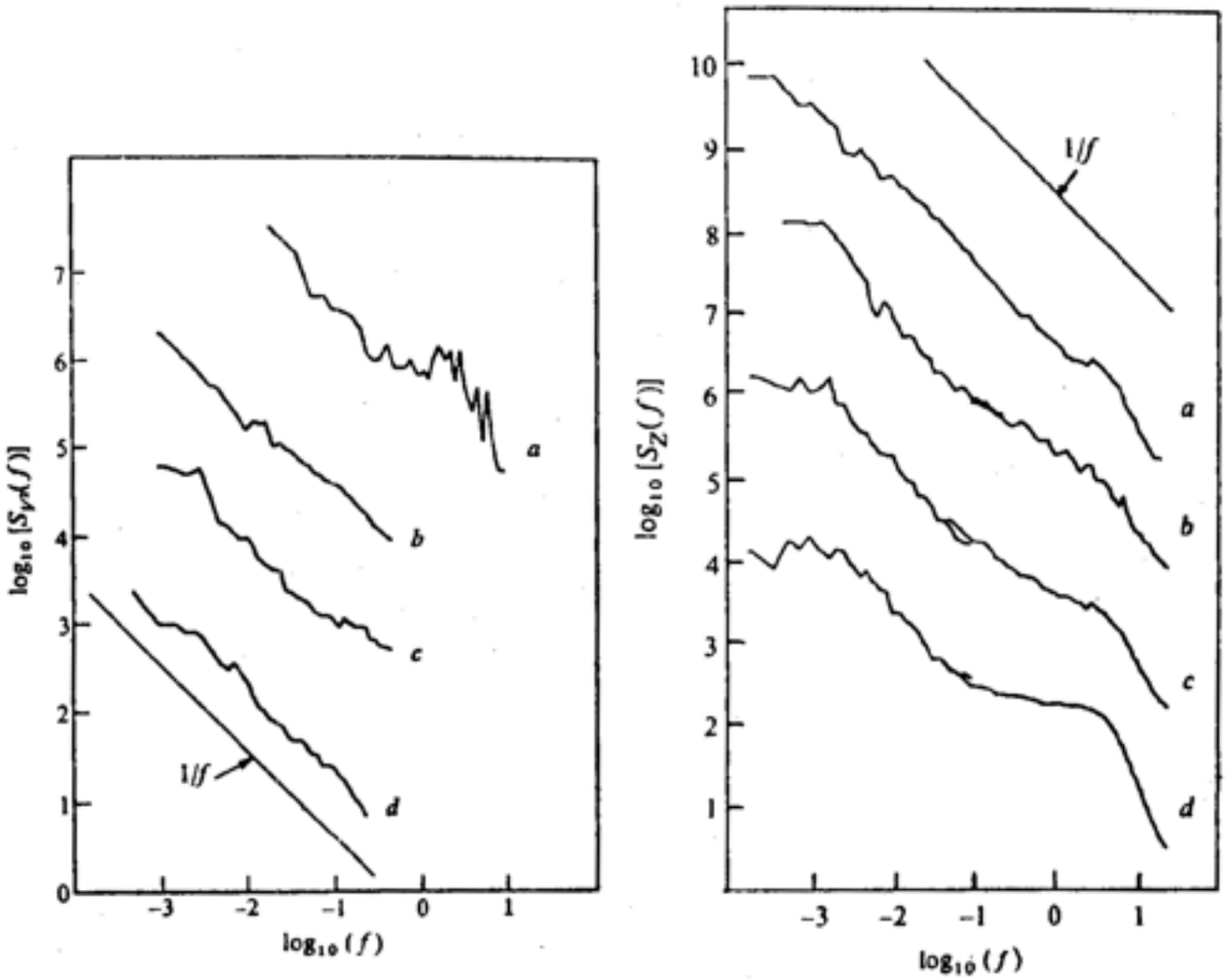
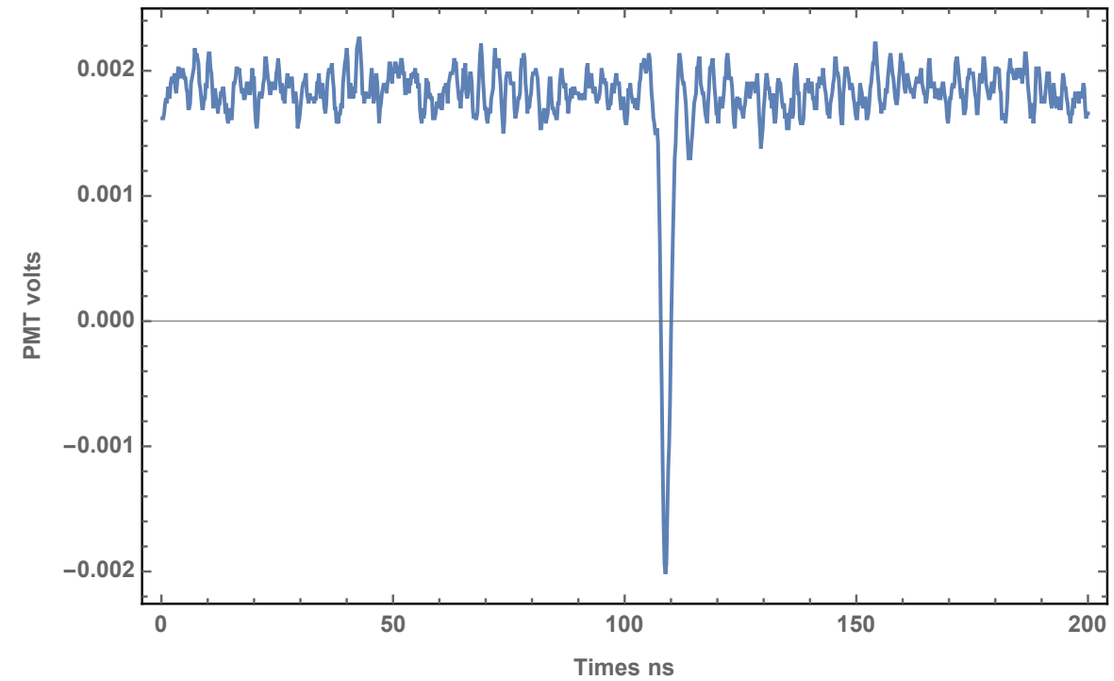
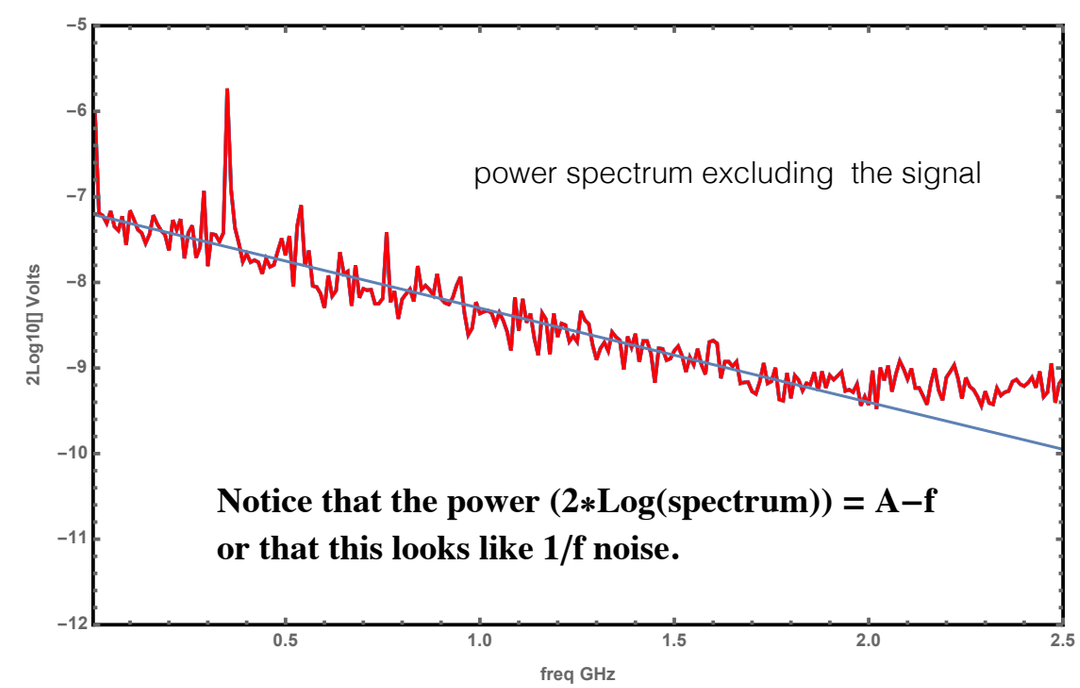


FIGURE 9 (*left*) Loudness fluctuation spectra as a function of frequency f (in Hz) for: *a*, Scott Joplin Piano Rags; *b*, classical radio station; *c*, rock station; *d*, news-and-talk station. (Reproduced from Voss and Clarke.[23]) (*right*) Power spectra of pitch fluctuations for four radio stations: *a*, classical; *b*, jazz and blues; *c*, rock; *d*, news-and-talk. (Reproduced from Voss and Clarke (1975).)

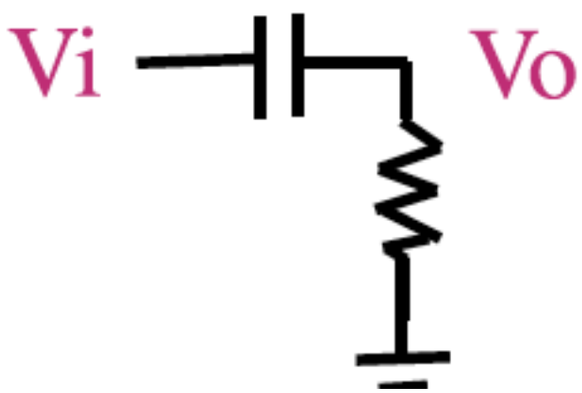




This noise is deeply related to the structure of the device and therefore could be sensitive to it failure modes.

Measurement of light pulses from a PMT by Aiwu Zhang

High pass, differentiator

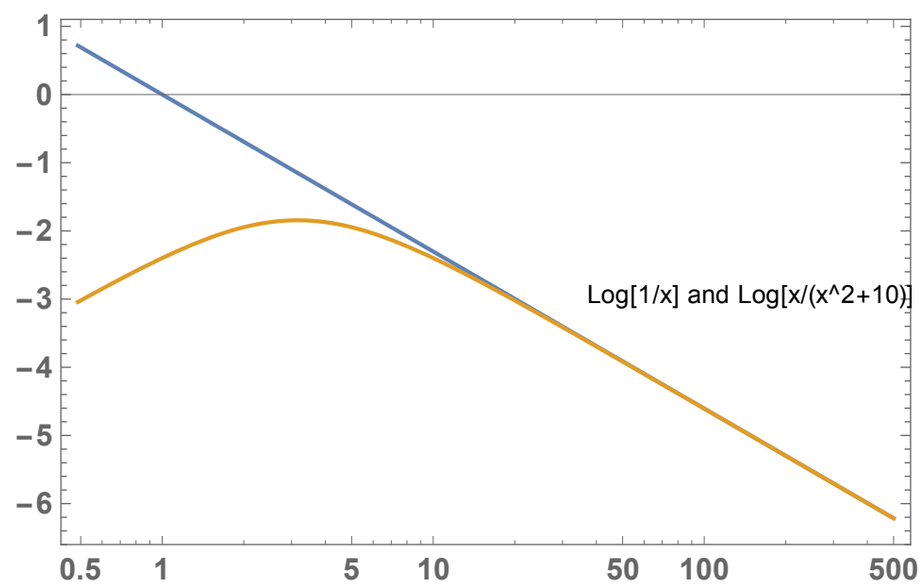


With the application of a high pass filter with time constant τ the 1/f noise can be relaxed so that it does not diverge at low frequency.

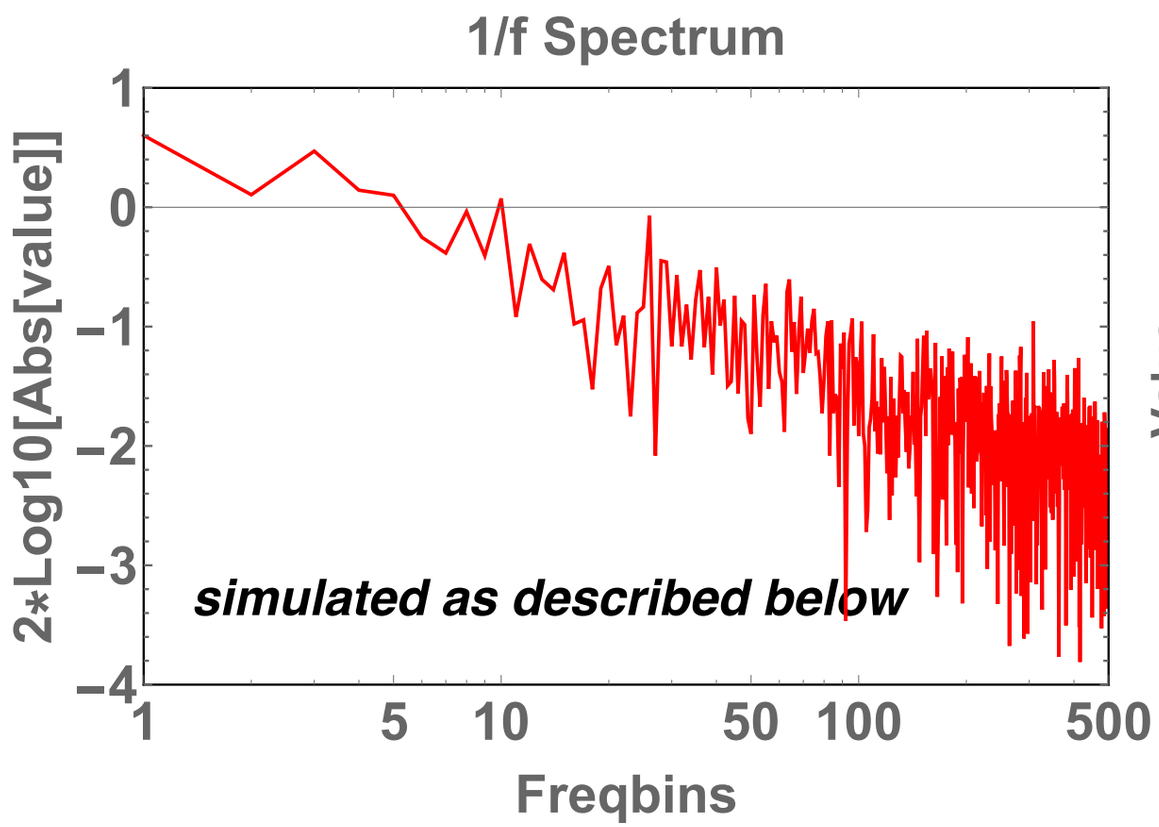
$$S_{Flick}(\omega)=\frac{1}{2\pi}\left[\frac{|G(\omega)|^2}{\omega}\times\frac{\omega^2}{(\omega^2+1/\tau^2)}\right]$$

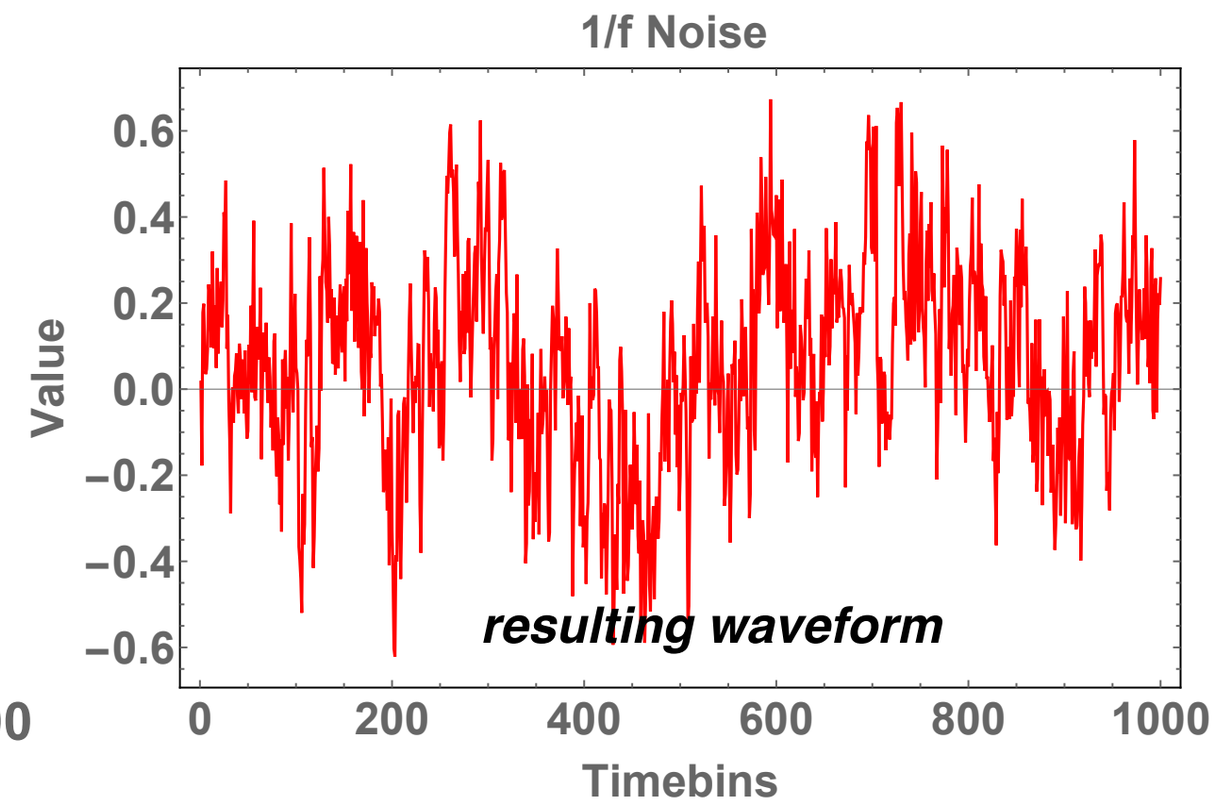
$$S_{Flick}(\omega)=\frac{1}{2\pi}\left[\frac{|G(\omega)|^2\times\omega}{(\omega^2+1/\tau^2)}\right]$$

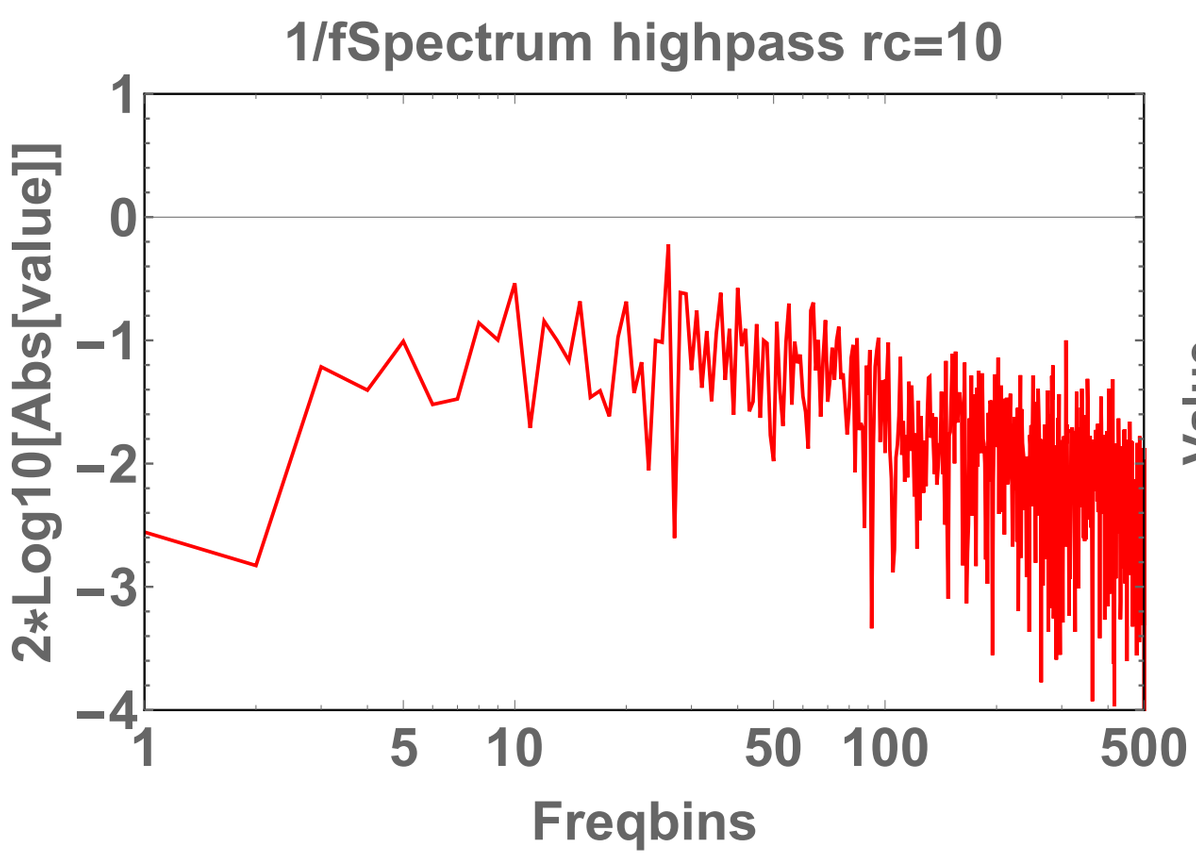
If $G(\omega)=1$ then the filter gives 0 for DC and rises for $\omega<<1/\tau$ and falls as $1/\omega$ for $\omega>>1/\tau$

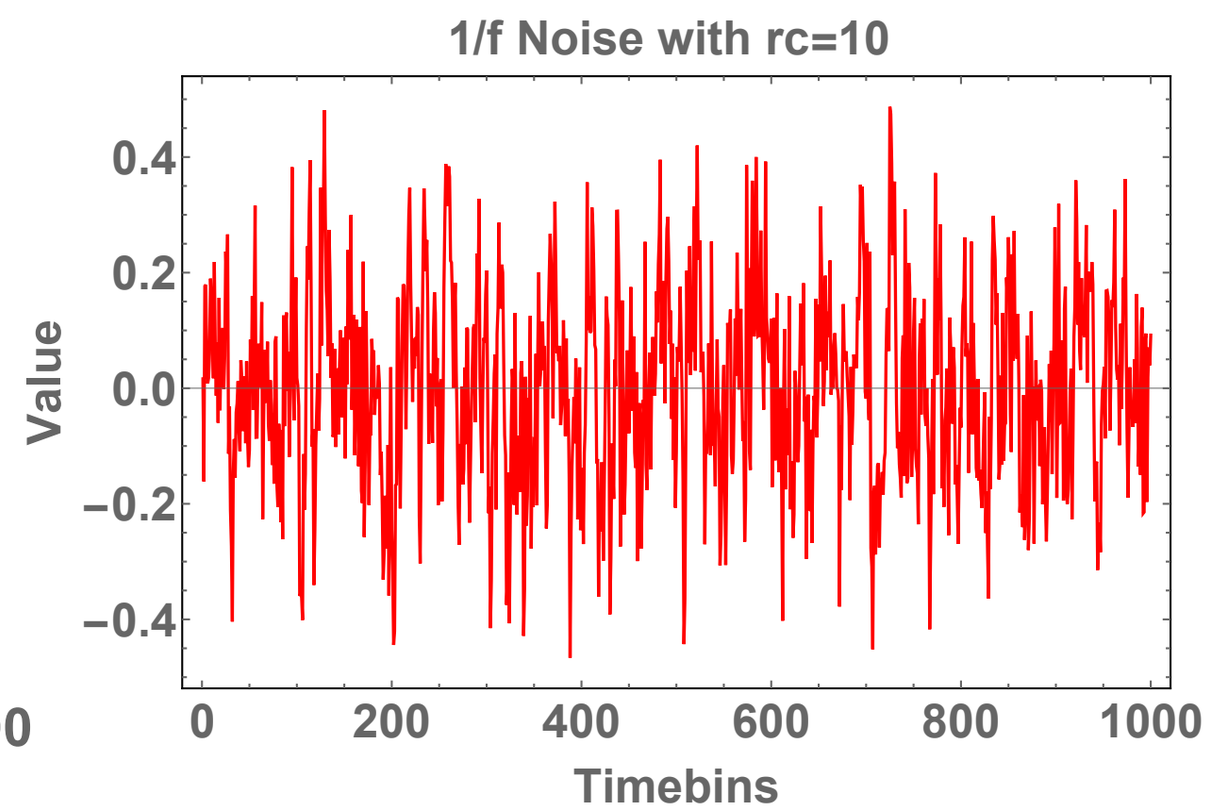


Notice that the power is zero for at f=0 regardless of the value of the time constant.


1/f Spectrum
2*Log10[Abs[value]]
1
0
−1
−2
−3
−4
1
5
10
50
100
500
Freqbins
simulated as described below


1/f Noise
Value
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
0
200
400
600
800
1000
Timebins
resulting waveform


1/fSpectrum highpass rc=10
2*Log10[Abs[value]]
1
0
−1
−2
−3
−4
1
5
10
50
100
500
Freqbins


1/f Noise with rc=10
Value
0.4
0.2
0.0
−0.2
−0.4
0
200
400
600
800
1000
Timebins

Finish part 2.

- ***We derived the Gaussian nature of most noises by using the Central Limit Theorem.***
- ***In general, there is a class of "stable" PDFs that result from the CLT. (e.g. Landau PDF)***
- ***We used the tool of elementary excitations and the delta function, but the results are more general.***
- ***The tools of using delta functions actually can be easily extended to perform Monte Carlo calculations.***
- ***Classification of noise spectra and their origin.***