ForWaRD: Fourier-Wavelet Regularized Deconvolution

Proposed extension to WC signal processing

Brett Viren

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Topics

- Problem statement
- Brief intro to general and Fourier signal basis
- Naive solution
- FoRD: Fourier Regularized Deconvolution (currently what WC uses)
- Problems with FoRD
- Brief intro to wavelets
- ForWaRD: Fourier-Wavelet Regularized Deconvolution
- Application of FoRD and ForWaRD to toy 1D LArTPC sim+sigproc
- Toward full implementation.

Measure a signal in the presence of noise

Linear discrete system

$$y[n] = \mathcal{H}x[n] + \gamma[n]$$

- $x[n]$ **signal** (unknown) as function of discrete time coordinate $n \in [0, ..., N-1]$.
- $\gamma[n]$ **noise**
- $y[n]$ **measurement**
- $\mathcal{H}$ **detector response**:
  - Assume linear convolution model: $\mathcal{H}x[n] \equiv (h * x)[n]$, $h[n]$ is **impulse response** of H.

Problem: find good estimate of the signal

$$\tilde{x}[n] \approx x[n]$$

Orthonormal vector basis: $\{\psi_k\}$ on $\mathbb{R}^N$ and its **dual set** $\{\tilde{\psi}_k\}$.

Decomposition of **sampled signal vector** $x[n],\ n \in [0, N-1]$ onto basis vectors

$$x[n] = \sum_k X[k]\psi_k[n]$$

Representation of signal as vector of coefficients (dual-basis dot product)

$$X[k] \equiv \sum_{n=0}^{N-1} \tilde{\psi}_k^*[n]x[n] \equiv < \tilde{\psi}_k, x >$$

Reconstruction of vector in original basis

$$x[n] = \sum_{k=0}^{N-1} X[k]\psi_k[n]$$

The Fourier basis

Fourier vector basis and its dual in terms of "N^{th} root of unity" z_N

$$\psi_k[n] = \frac{1}{N}(z_N)^{kn},\ \tilde{\psi}_k[n] \equiv (z_N)^{kn},\ z_N \equiv e^{i2\pi/N}$$

Convolution

$$(a * b)[n] \leftrightarrow A[k]B[k]$$

Parseval energy equality

$$\sum_n |x[n]|^2 = \frac{1}{N}\sum_k |X[k]|^2$$

Aside: generating a basis from a "mother"

Euclidean $\mathbb{R}^N$ (*eg*, time domain) basis from Dirac-δ

$$\psi \equiv \delta_{0,n},\ shift(\psi_0, k) \rightarrow \delta_{k,n},\ \{\psi_k\} = \{\delta_{k,n}\}$$

Fourier basis from "N^{th} root of unity" z_N

$$\psi_k[n] = \frac{1}{N}(z_N)^{kn},\ \tilde{\psi}_k[n] \equiv (z_N)^{kn},\ z_N \equiv e^{i2\pi/N}$$

Wavelet basis from mother wavelet + scaling functions

We will get to this.

The “naive” estimate

Back to solving for x in $y[n] = \mathcal{H}x[n] + \gamma[n]$

Apply inverse detector response $\mathcal{H}^{-}1$ in Fourier basis for estimate $\tilde{x}$

$$\tilde{x}[n] \equiv \mathcal{H}^{-1}y[n] = x[n] + \mathcal{H}^{-1}\gamma[n]$$

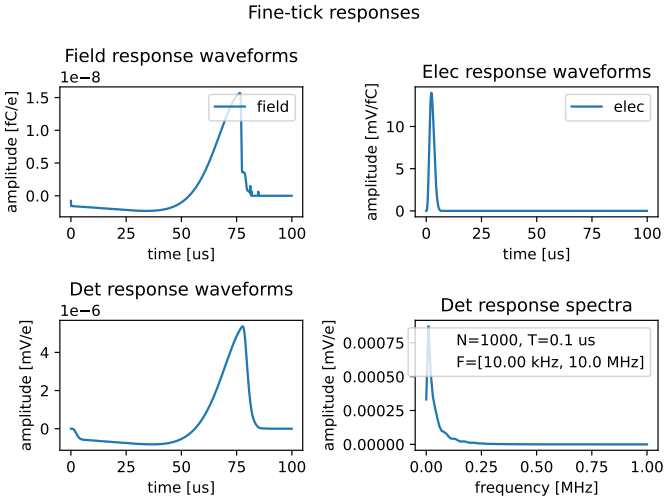
$$\tilde{X}[k] = \frac{Y[k]}{H[k]} = X[k] + \frac{\Gamma[k]}{H[k]}$$

Why is it called “naive”?

- Influence of noise Γ remains and it will be **amplified** when H is small (or even zero).

Next, look at $\mathcal{H}$ and γ for real detectors....

Impulse response $\mathcal{H}$ for MicroBooNE



Waveform:

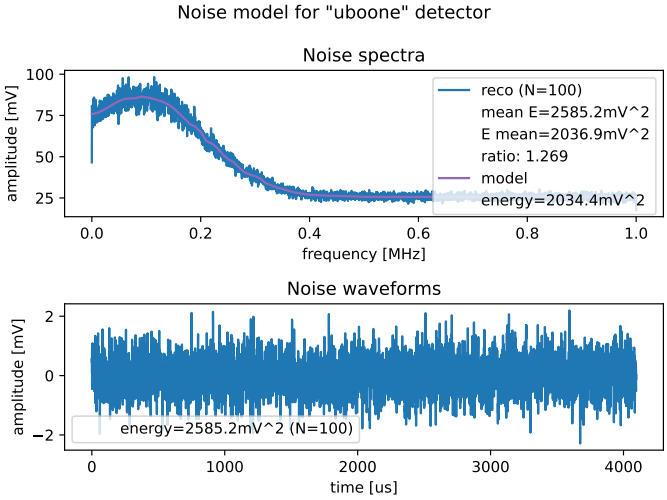
- U-plane, bipolar, asymmetric.

Spectrum:

- Strongly peaked at low frequency
- Sharp dive at $k = 0$ frequency
  - Should go to zero but calculation domain was too small.
- Strong attenuation above ≈ 200 kHz

Noise spectrum for MicroBooNE

Mean spectrum model → generate waveforms → reconstruct mean spectrum.



Noise is substantial across a broad band.

- Particularly a $k \approx 0$ where H is small

Note: need care in **mean** *vs* **mode** when generating waveforms and applying Parseval energy.

Toy signal model

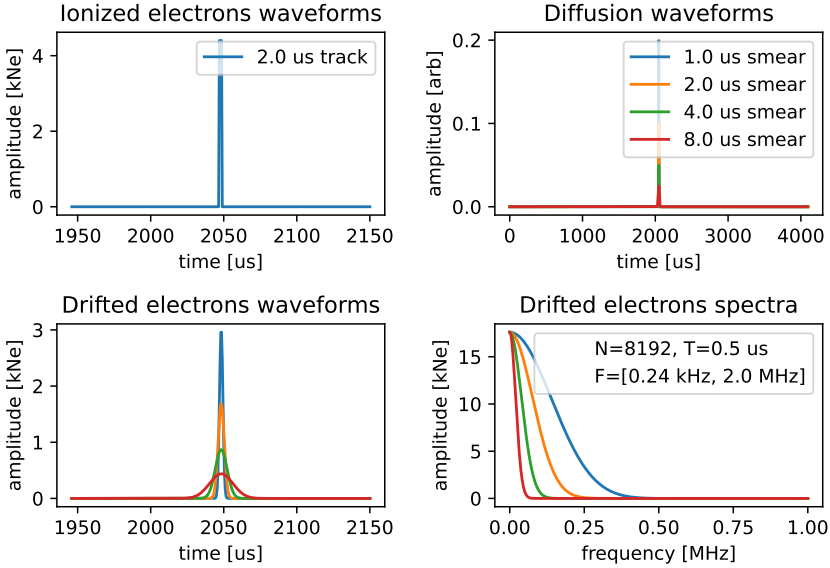
“Detector”

- quasi-1D field response integrating 2D response over central wire
- simple Gaussian smearing for diffusion.
- look at voltage level, no ADC

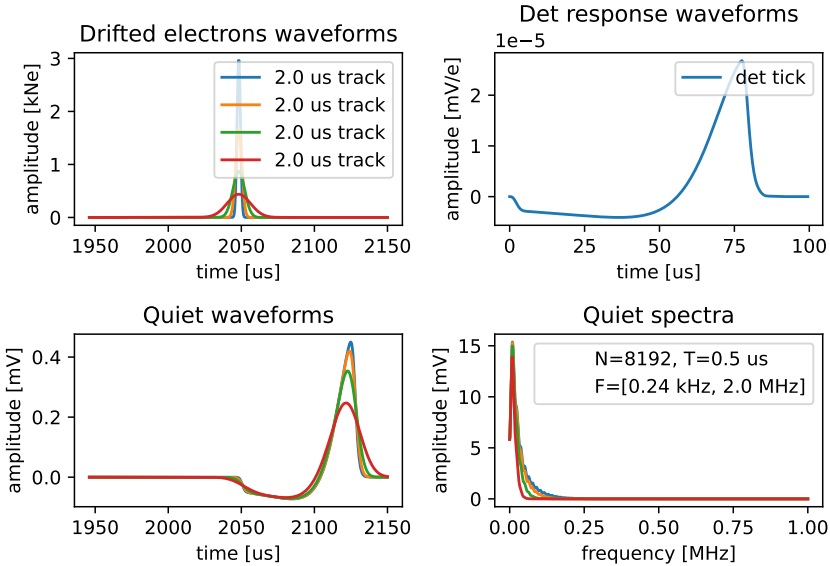
“Kinematics”

- Ideal “track” as square wave in time with MIP-level electron density.
- Slightly less ideal “blips” uniformly random Gaussians filling a square wave.

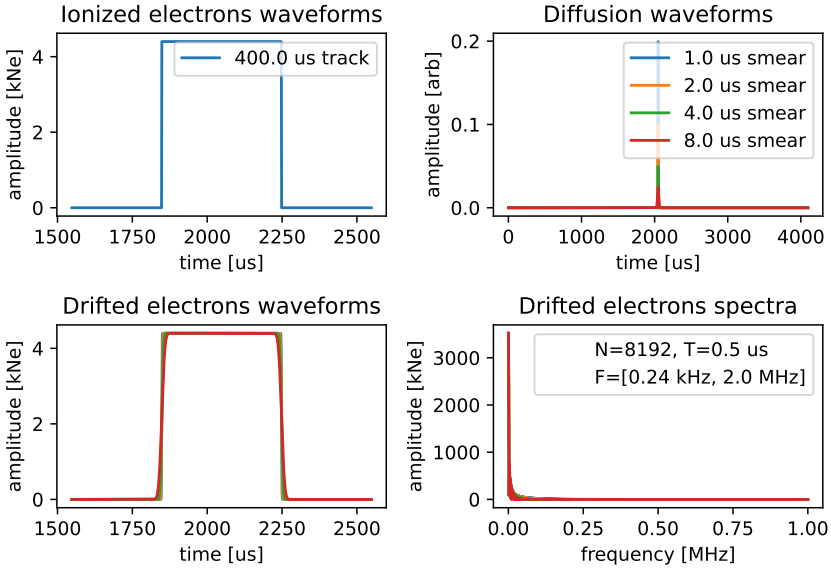
Ideal track ionization and diffusion



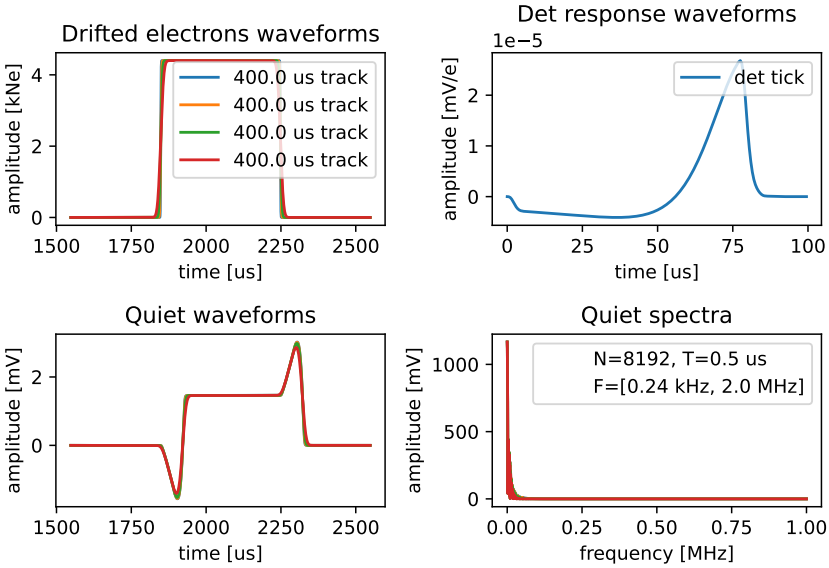
Ideal track measured signal



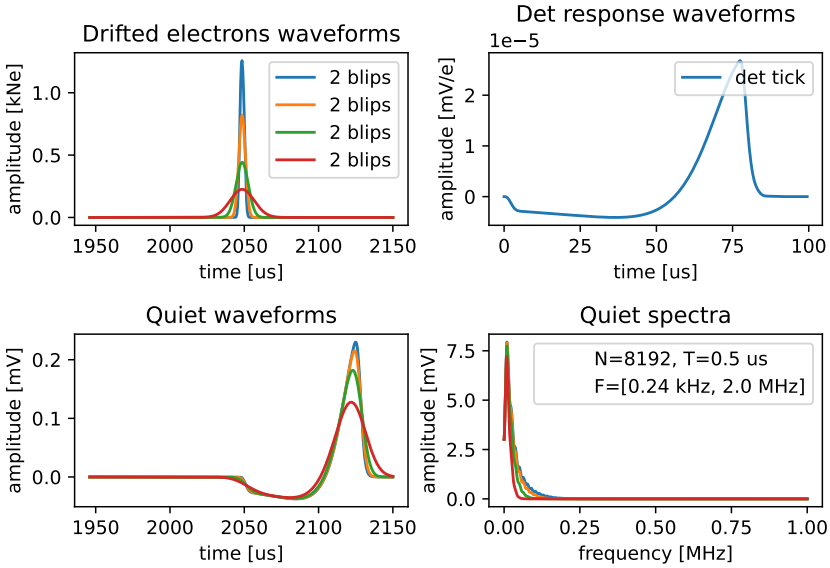
Ideal track ionization and diffusion



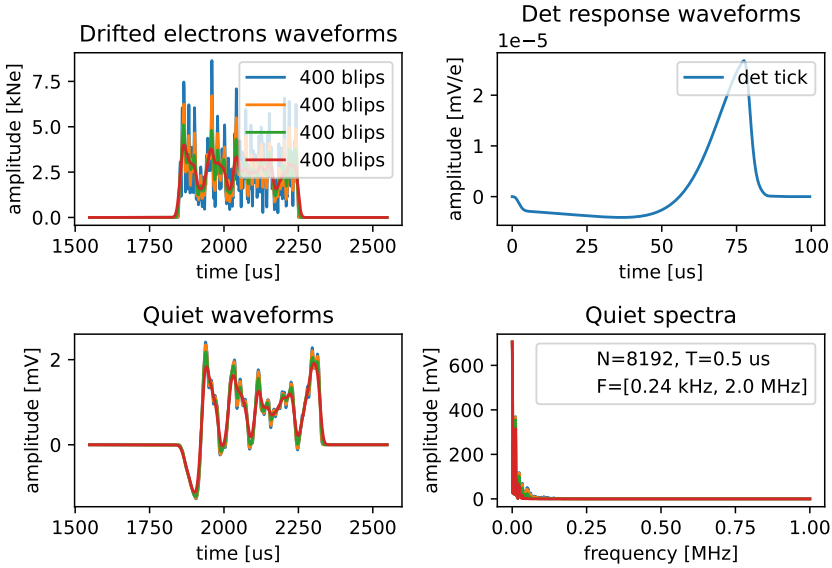
Ideal track measured signal



Blips measured signal



Blips measured signal



Protect against divide-by-zero

Modify the naive estimate in obvious way

$$\tilde{X}[k] \equiv \begin{cases} X[k] + \frac{\Gamma[k]}{H[k]} & \text{if } H[k] > 0 \\ 0 & \text{otherwise} \end{cases}$$

That is, we **destroy** all noise at any frequency where response is zero.

- This idea is extended next to provide a **shink** some frequencies instead of destroy them.

FoRD: Fourier Regularized Deconvolution (WC's "SP filtering")

Introduce **Fourier shrinkage** coefficients $\lambda^f[k]$ with **regularization terms** $\Lambda[k]$

$$\lambda^f[k] = \frac{|H[k]|^2}{|H[k]|^2 + \Lambda[k]},\ \Lambda[k] \geq 0$$

DFT of **FoRD estimate** $\tilde{X}_{\lambda^f}$

$$\tilde{X}_{\lambda^f}[k] \equiv \tilde{X}[k]\lambda^f[k] \equiv X_{\lambda^f}[k] + \frac{\Gamma_{\lambda^f}[k]}{H[k]}$$

- Attenuate ("shink") the amplified noise in naive estimate $\tilde{X}$ when $\Lambda > 0,\ \lambda_f < 1$
  - And may **distort the signal**
- Define terms: DFT of **retained signal** X_{λ^f} and DFT of **leaked noise** $\frac{\Gamma_{\lambda^f}[k]}{H[k]}$

FoRD: defining shrinkage coefficients

Wiener regularization term

$$\Lambda[k] \equiv \alpha \frac{|\Gamma[k]|^2}{|X[k]|^2} \rightarrow \lambda^f[k] = \frac{|Y[k]|^2}{|Y[k]|^2 + \alpha|\Gamma[k]|^2}$$

Optimize a single parameter α across all frequencies (and events).

- $\alpha = 0$ naive deconvolution.
- $\alpha \approx 1$ more strongly shrink noisy frequencies.

Other forms

- Tikhonov: $\Lambda[k] \equiv \tau$.
- Allpass: $\lambda[k] = |H[k]|$

FoRD shrinkage vs WCT filtering

FoRD

A single Wiener filter with **fixed noise** model.

- Ends up being naturally peaked in frequency (**band pass**).
- May be **adaptive**, based on each **measured** waveform or **fixed** based on some average signal.

WCT

Pair of **fixed** filters based on average **simulated signal** and **fixed noise model**.

- "Wiener-inspired" **low-pass** filter provides "half" Wiener filter.
- Add one of two ad-hoc **high-pass** ("low-frequency") filters to form **band pass**.
  - "tight" low noise, attenuates long signals.
  - "loose" higher noise, retains long signals.

FoRD: signal and noise

FoRD: pros and cons

Brief intro to wavelets

A wavelet basis is **generated** from:

- A "mother" wavelet function $\psi[n]$
- A scaling function $\phi[n]$
- A time-shift operator τ_l

Similar to "normal" basis

- Euclidean basis fully localized in time, generated by a shift operator

ForWaRD: Fourier-Wavelet Regularized Deconvolution

Application of FoRD and ForWaRD to toy 1D LArTPC sim+sigproc

Toward full implementation in Wire-Cell toolkit