Hadronic Phenomenology from AdS/CFT for Experimentalists
Applications of AdS/CFT to QCD

Changes in physical length scale mapped to evolution in the 5th dimension $z$

in collaboration with Guy de Teramond

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AdS/QCD

Stan Brodsky, SLAC
Goal:

• Use AdS/CFT to provide an approximate, covariant, and analytic model of hadron structure with confinement at large distances, conformal behavior at short distances

• Analogous to the Schrodinger Coulomb Equation for Atomic Physics

• AdS/QCD Holographic Model
\[ M^2 = 2\kappa^2(2n + 2L + S). \]

\[ S = 1 \]
\[
\left[-\frac{d^2}{d\zeta^2} + V(\zeta)\right] \phi(\zeta) = M^2 \phi(\zeta) \\
S = 0
\]

\[
V(\zeta) = -\frac{1 - 4L^2}{4\zeta^2} + \kappa^4 \zeta^2
\]

\[
\zeta^2 = x(1 - x)b_\perp^2.
\]

\[\psi_\pi(x, b_\perp)\]

\textbf{Prediction from AdS/CFT: Meson LFWF}
AdS/CFT: Anti-de Sitter Space / Conformal Field Theory

Maldacena:

Map $\text{AdS}_5 \times S_5$ to conformal $N=4$ SUSY

- **QCD is not conformal**: however, it has manifestations of a scale-invariant theory: Bjorken scaling, dimensional counting for hard exclusive processes

- **Conformal window**: $\alpha_s(Q^2) \sim \text{const at small } Q^2$

- **Use mathematical mapping of the conformal group $SO(4,2)$ to AdS5 space**
Concurrent Counting Rules

\[ n_{tot} = n_A + n_B + n_C + n_D \]

\[ \frac{d\sigma}{dt}(s, t) = \frac{F(\theta_{cm})}{s[n_{tot} - 2]} \quad s = E_{cm}^2 \]

\[ F_H(Q^2) \sim \left[ \frac{1}{Q^2} \right]^{n_H - 1} \]

Conformal symmetry and PQCD predict leading-twist scaling behavior of fixed-CM angle exclusive amplitudes

Farrar & sjb; Matveev, Muradyan, Tavkhelidze
Quark-Counting: \[ \frac{d\sigma}{dt}(pp \rightarrow pp) = \frac{F(\theta_{CM})}{s^{10}} \]

\[ n = 4 \times 3 - 2 = 10 \]

**Best Fit**

\[ n = 9.7 \pm 0.5 \]

Reflects underlying conformal scale-free interactions

**Angular distribution -- quark interchange**

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• Phenomenological success of dimensional scaling laws for exclusive processes

$$d\sigma/dt \sim 1/s^{n-2}, \quad n = n_A + n_B + n_C + n_D,$$

implies QCD is a strongly coupled conformal theory at moderate but not asymptotic energies 
Farrar and sjb (1973); Matveev et al. (1973).

• Derivation of counting rules for gauge theories with mass gap dual to string theories in warped space
Conformal QCD Window in Exclusive Processes

- Does $\alpha_s$ develop an IR fixed point? Dyson–Schwinger Equation Alkofer, Fischer, LLanes-Estrada, Deur …

- Recent lattice simulations: evidence that $\alpha_s$ becomes constant and is not small in the infrared
  Furui and Nakajima, hep-lat/0612009  (Green dashed curve: DSE).
\[ \Gamma_{bj}^{p-n}(Q^2) \equiv \frac{g_A}{6} \left[ 1 - \frac{\alpha_s g_1(Q^2)}{\pi} \right] \]
New Perspectives on QCD Phenomena from AdS/CFT

• **AdS/CFT**: Duality between string theory in Anti-de Sitter Space and Conformal Field Theory

• New Way to Implement Conformal Symmetry

• Holographic Model: Conformal Symmetry at Short Distances, Confinement at large distances

• Remarkable predictions for hadronic spectra, wavefunctions, interactions

• AdS/CFT provides novel insights into the quark structure of hadrons
Conformal symmetry: Template for QCD

- Take conformal symmetry as initial approximation; then correct for non-zero beta function, confinement, and quark masses
- Eigensolutions of ERBL evolution equation for distribution amplitudes
- Commensurate scale relations: relate observables at corresponding scales: Generalized Crewther Relation
- Fix Renormalization Scale (BLM)
- Use AdS/CFT
Conformal Theories are invariant under the Poincare and conformal transformations with

\[ M^{\mu\nu}, P^\mu, D, K^\mu, \]

the generators of \( \text{SO}(4,2) \)

\( \text{SO}(4,2) \) has a mathematical representation on AdS5
Scale Transformations

- Isomorphism of $SO(4,2)$ of conformal QCD with the group of isometries of AdS space.

$$ds^2 = \frac{R^2}{z^2} (\eta_{\mu\nu} dx^{\mu} dx^{\nu} - dz^2),$$

$$x^\mu \rightarrow \lambda x^\mu, \ z \rightarrow \lambda z,$$ maps scale transformations into the holographic coordinate $z$.

- AdS mode in $z$ is the extension of the hadron wf into the fifth dimension.

- Different values of $z$ correspond to different scales at which the hadron is examined.

$$x^2 \rightarrow \lambda^2 x^2, \ z \rightarrow \lambda z.$$  

$x^2 = x_\mu x^\mu$: invariant separation between quarks

- The AdS boundary at $z \rightarrow 0$ correspond to the $Q \rightarrow \infty$, UV zero separation limit.
We will consider both holographic models

- Truncated AdS/CFT (Hard-Wall) model: cut-off at \( z_0 = 1/\Lambda_{QCD} \) breaks conformal invariance and allows the introduction of the QCD scale (Hard-Wall Model) Polchinski and Strassler (2001).

- Smooth cutoff: introduction of a background dilaton field \( \varphi(z) \) – usual linear Regge dependence can be obtained (Soft-Wall Model) Karch, Katz, Son and Stephanov (2006).

We will consider both holographic models

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• **Polchinski & Strassler**: AdS/CFT builds in conformal symmetry at short distances; counting rules for form factors and hard exclusive processes; non-perturbative derivation

• **Goal**: Use AdS/CFT to provide an approximate model of hadron structure with confinement at large distances, conformal behavior at short distances

• **de Teramond, sjb**: AdS/QCD Holographic Model: Initial “semi-classical” approximation to QCD. Predict light-quark hadron spectroscopy, form factors.

• **Karch, Katz, Son, Stephanov**: Linear Confinement

• Mapping of AdS amplitudes to 3+1 Light-Front equations, wavefunctions

• Use AdS/CFT wavefunctions as expansion basis for diagonalizing $H_{L\text{FQCD}}$; variational methods
Use mapping of conformal group SO(4,2) to AdS5

Scale Transformations represented by wavefunction $\psi(z)$ in 5th dimension

$$x_\mu^2 \rightarrow \lambda^2 x_\mu^2 \quad z \rightarrow \lambda z$$

Hard wall model: Confinement at large distances and conformal symmetry in interior $0 < z < z_0$

Match solutions at small $z$ to conformal dimension of hadron wavefunction at short distances $\psi(z) \sim z^\Delta$ at $z \rightarrow 0$

Truncated space simulates “bag” boundary conditions

$$\psi(z_0) = 0 \quad z_0 = \frac{1}{\Lambda_{QCD}}$$
**Match fall-off at small z to conformal twist-dimension at short distances**

- Pseudoscalar mesons: $\mathcal{O}_{2+L} = \overline{\psi} \gamma_5 D \{ \ell_1 \ldots D \ell_m \} \psi$ (\(\Phi_\mu = 0\) gauge). \(\Delta = 2 + L\)

- 4-\(d\) mass spectrum from boundary conditions on the normalizable string modes at \(z = z_0\), \(\Phi(x, z_0) = 0\), given by the zeros of Bessel functions \(\beta_{\alpha, k}\): 
  
  \[
  \mathcal{M}_{\alpha, k} = \beta_{\alpha, k} \Lambda_{QCD}
  \]

- Normalizable AdS modes \(\Phi(z)\)

\[ S = 0 \quad \text{Meson orbital and radial AdS modes for} \quad \Lambda_{QCD} = 0.32 \text{ GeV.} \]

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Bosonic Solutions: Hard Wall Model

- Conformal metric: \( ds^2 = g_{\ell m} dx^\ell dx^m \). \( x^\ell = (x^\mu, z) \), \( g_{\ell m} \rightarrow (R^2/z^2) \eta_{\ell m} \).

- Action for massive scalar modes on AdS\(_{d+1}\):

\[
S[\Phi] = \frac{1}{2} \int d^{d+1}x \sqrt{g} \frac{1}{2} \left[ g^{\ell m} \partial_\ell \Phi \partial_m \Phi - \mu^2 \Phi^2 \right], \quad \sqrt{g} \rightarrow (R/z)^{d+1}.
\]

- Equation of motion

\[
\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\ell} \left( \sqrt{g} g^{\ell m} \frac{\partial}{\partial x^m} \Phi \right) + \mu^2 \Phi = 0.
\]

- Factor out dependence along \( x^\mu \)-coordinates, \( \Phi_P(x, z) = e^{-iP \cdot x} \Phi(z) \), \( P_\mu P^\mu = \mathcal{M}^2 \):

\[
\left[ z^2 \partial_z^2 - (d - 1)z \partial_z + z^2 \mathcal{M}^2 - (\mu R)^2 \right] \Phi(z) = 0.
\]

- Solution: \( \Phi(z) \rightarrow z^\Delta \) as \( z \rightarrow 0 \),

\[
\Phi(z) = C z^{d/2} J_{\Delta - d/2}(z \mathcal{M})
\]

\[
\Delta = 2 + L \quad d = 4 \quad (\mu R)^2 = L^2 - 4
\]

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AdS Schrödinger Equation for bound state of two scalar constituents:

\[
\left[-\frac{d^2}{dz^2} + V(z)\right] \phi(z) = M^2 \phi(z)
\]

\[V(z) = -\frac{1-4L^2}{4z^2}\]

Derived from variation of Action in AdS$_5$

Hard wall model: truncated space

\[\phi(z = z_0 = \frac{1}{\Lambda_c}) = 0.\]
AdS Schrödinger Equation for bound state of two scalar constituents:

\[
\left[ -\frac{d^2}{dz^2} + V(z) \right] \phi(z) = M^2 \phi(z)
\]

**Hard wall model:** truncated space

\[
V(z) = -\frac{1 - 4L^2}{4z^2} \quad \phi(z = z_0 = 1/\Lambda_0) = 0
\]

**Soft wall model:** Harmonic oscillator confinement

\[
V(z) = -\frac{1 - 4L^2}{4z^2} + \kappa^4 z^2
\]

Derived from variation of Action in AdS$_5$
Orbital and radial AdS modes in the hard wall model for $\Lambda_{QCD} = 0.32$ GeV.

Light meson and vector meson orbital spectrum $\Lambda_{QCD} = 0.32$ GeV.
Fig: Orbital and radial AdS modes in the soft wall model for $\kappa = 0.6$ GeV.

Light meson orbital (a) and radial (b) spectrum for $\kappa = 0.6$ GeV.
Higher Spin Bosonic Modes SW

- Effective LF Schrödinger wave equation
\[
\left[ -\frac{d^2}{d\zeta^2} - \frac{1 - 4L^2}{4\zeta^2} + \kappa^4 \zeta^2 + 2\kappa^2 (L + S - 1) \right] \phi_S(\zeta) = \mathcal{M}^2 \phi_S(\zeta)
\]
with eigenvalues \( \mathcal{M}^2 = 2\kappa^2 (2n + 2L + S) \).

- Compare with Nambu string result (rotating flux tube):
\[ M_n^2(L) = 2\pi \sigma (n + L + 1/2) \] .

- Glueballs in the bottom-up approach: (HW) Boschi-Filho, Braga and Carrion (2005); (SW) Colangelo, De Facio, Jugeau and Nicotri (2007).

Vector mesons orbital (a) and radial (b) spectrum for \( \kappa = 0.54 \text{ GeV} \).

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\[ \alpha(t) \approx \frac{1}{2} + 0.9t \]

**AdS/QCD Soft Wall Model -- Reproduces Linear Regge Trajectories**
Hadron Form Factors from AdS/CFT

Propagation of external perturbation suppressed inside AdS.

\[ J(Q, z) = z Q K_1(zQ) \]

\[ F(Q^2)_{I \rightarrow F} = \int \frac{dz}{z^3} \Phi_F(z) J(Q, z) \Phi_I(z) \]

High \( Q^2 \) from small \( z \sim 1/Q \)

Polchinski, Strassler
de Teramond, sjb

Consider a specific AdS mode \( \Phi^{(n)} \) dual to an \( n \) partonic Fock state \( |n\rangle \). At small \( z \), \( \Phi^{(n)} \) scales as \( \Phi^{(n)} \sim z^{\Delta_n} \). Thus:

\[ F(Q^2) \rightarrow \left[ \frac{1}{Q^2} \right]^{\tau-1} \]

where \( \tau = \Delta_n - \sigma_n \), \( \sigma_n = \sum_{i=1}^{n} \sigma_i \). The twist is equal to the number of partons, \( \tau = n \).

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Spacelike pion form factor from AdS/CFT

\[ F_\pi(q^2) \]

Data Compilation from Baldini, Kloe and Volmer

SW: Harmonic Oscillator Confinement

HW: Truncated Space Confinement

One parameter - set by pion decay constant.

dede Teramond, sjb

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Note: Contributions to Mesons Form Factors at Large $Q$ in AdS/QCD

- Write form factor in terms of an effective partonic transverse density in impact space $b_\perp$

$$F_\pi(q^2) = \int_0^1 dx \int d^2b \tilde{\rho}(x, b, Q),$$

with $\tilde{\rho}(x, b, Q) = \pi J_0 [b Q (1 - x)] |\tilde{\psi}(x, b)|^2$ and $b = |b_\perp|$.

- Contribution from $\rho(x, b, Q)$ is shifted towards small $|b_\perp|$ and large $x \to 1$ as $Q$ increases.

Fig: LF partonic density $\rho(x, b, Q)$: (a) $Q = 1$ GeV/c, (b) very large $Q$. 

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Note: Analytical Form of Hadronic Form Factor for Arbitrary Twist

- Form factor for a string mode with scaling dimension $\tau$, $\Phi_\tau$ in the SW model

$$F(Q^2) = \Gamma(\tau) \frac{\Gamma \left(1 + \frac{Q^2}{4\kappa^2}\right)}{\Gamma \left(\tau + \frac{Q^2}{4\kappa^2}\right)}.$$ 

- For $\tau = N$, $\Gamma(N + z) = (N - 1 + z)(N - 2 + z) \ldots (1 + z)\Gamma(1 + z)$.

- Form factor expressed as $N - 1$ product of poles

$$F(Q^2) = \frac{1}{1 + \frac{Q^2}{4\kappa^2}}, \quad N = 2,$$
$$F(Q^2) = \frac{2 \left(1 + \frac{Q^2}{4\kappa^2}\right) \left(2 + \frac{Q^2}{4\kappa^2}\right)}{\ldots}, \quad N = 3,$$
$$F(Q^2) = \frac{(N - 1)!}{\left(1 + \frac{Q^2}{4\kappa^2}\right) \left(2 + \frac{Q^2}{4\kappa^2}\right) \ldots \left(N - 1 + \frac{Q^2}{4\kappa^2}\right)}, \quad N.$$

- For large $Q^2$:

$$F(Q^2) \rightarrow (N - 1)! \left[\frac{4\kappa^2}{Q^2}\right]^{(N-1)}.$$
Light-Front Representation of Two-Body Meson Form Factor

• Drell-Yan-West form factor

\[ F(q^2) = \sum_q e_q \int_0^1 dx \int \frac{d^2 k_\perp}{16\pi^3} \psi^*_P(x, k_\perp - xq_\perp) \psi_P(x, k_\perp). \]

• Fourier transform to impact parameter space \( \vec{b}_\perp \)

\[ \psi(x, k_\perp) = \sqrt{4\pi} \int d^2 \vec{b}_\perp e^{i\vec{b}_\perp \cdot \vec{k}_\perp} \tilde{\psi}(x, \vec{b}_\perp) \]

• Find \( (b = |\vec{b}_\perp|) : \)

\[ F(q^2) = \int_0^1 dx \int d^2 \vec{b}_\perp e^{ix\vec{b}_\perp \cdot \vec{q}_\perp} |\tilde{\psi}(x, b)|^2 \]

\[ = 2\pi \int_0^1 dx \int_0^\infty b \, db \, J_0(bqx) \left| \tilde{\psi}(x, b) \right|^2, \]

Soper

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Holographic Mapping of AdS Modes to QCD LFWFs

- Integrate Soper formula over angles:

\[ F(q^2) = 2\pi \int_0^1 dx \frac{(1 - x)}{x} \int \zeta d\zeta J_0 \left( \zeta q \sqrt{\frac{1 - x}{x}} \right) \tilde{\rho}(x, \zeta), \]

with \( \tilde{\rho}(x, \zeta) \) QCD effective transverse charge density.

- Transversality variable

\[ \zeta = \sqrt{\frac{x}{1 - x}} \left| \sum_{j=1}^{n-1} x_j b_{\perp j} \right|. \]

- Compare AdS and QCD expressions of FFs for arbitrary \( Q \) using identity:

\[ \int_0^1 dx J_0 \left( \zeta Q \sqrt{\frac{1 - x}{x}} \right) = \zeta Q K_1(\zeta Q), \]

the solution for \( J(Q, \zeta) = \zeta Q K_1(\zeta Q) \)!
$\phi(z)$

$\zeta = \sqrt{x(1-x)b_\perp^2}$

$\psi(x, \zeta) = \sqrt{x(1-x)\zeta^{-1/2}} \phi(\zeta)$

**Holography:** Unique mapping derived from equality of LF and AdS formula for current matrix elements
Holography:
Map AdS/CFT to 3+1 LF Theory

Relativistic LF radial equation

\[
\left[-\frac{d^2}{d\zeta^2} + V(\zeta)\right] \phi(\zeta) = \mathcal{M}^2 \phi(\zeta)
\]

\[
\zeta^2 = x(1-x)b^2_\perp.
\]

Effective conformal potential:

\[
V(\zeta) = -\frac{1 - 4L^2}{4\zeta^2} + \kappa^4 \zeta^2.
\]

Confining potential:

\[
V(\zeta) = -\frac{1 - 4L^2}{4\zeta^2} + \kappa^4 \zeta^2.
\]

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Example: Pion LFWF

- Two parton LFWF bound state:

\[
\tilde{\psi}_{q\bar{q}/\pi}^{HW}(x, b_\perp) = \frac{\Lambda_{\text{QCD}} \sqrt{x(1-x)}}{\sqrt{\pi} J_{1+L}(b_\perp \beta_L, \kappa \Lambda_{\text{QCD}})} J_L \left( \sqrt{x(1-x)} |b_\perp| \beta_L, \kappa \Lambda_{\text{QCD}} \right) \theta \left( b_\perp^2 \leq \frac{\Lambda_{\text{QCD}}^2}{x(1-x)} \right),
\]

\[
\tilde{\psi}_{q\bar{q}/\pi}^{SW}(x, b_\perp) = \kappa^{L+1} \sqrt{\frac{2n!}{(n+L)!}} [x(1-x)]^{1/2+L} |b_\perp|^L e^{-\frac{1}{2} \kappa^2 x(1-x)} b_\perp^2 L_n^L (\kappa^2 x(1-x) b_\perp^2).
\]

Fig: Ground state pion LFWF in impact space. (a) HW model \( \Lambda_{\text{QCD}} = 0.32 \) GeV, (b) SW model \( \kappa = 0.375 \) GeV.
\[ \psi_M(x, k^2_\perp) \]

\[ \phi_M(x, Q_0) \propto \sqrt{x(1-x)} \]

\[ \psi_M(x, k^\perp) = \frac{4\pi}{\kappa \sqrt{x(1-x)}} e^{-\frac{k^2_\perp}{2\kappa^2 x(1-x)}} \]

\( \kappa = 0.375 \text{ GeV} \)

massless quarks

“Soft Wall” model

\text{de Teramond, sjb}
Hadron Distribution Amplitudes

- Fundamental gauge invariant non-perturbative input to hard exclusive processes, heavy hadron decays. Defined for mesons, baryons

- Evolution Equations from PQCD, OPE, Conformal Invariance

- Compute from valence light-front wavefunction in light-cone gauge

\[
\phi_H(x_i, Q) = \sum_i x_i = 1
\]

\[
\phi_M(x, Q) = \int^Q d^2 k \psi_{q\bar{q}}(x, k_\perp)
\]

\[k_\perp^2 < Q^2\]

\[1 - x\]

\[\text{Fixed } \tau = t + z/c\]

Lepage, sjb

Lepage, sjb, Frishman, Lepage, Sachrajda, sjb

Peskin Braun

Efremov, Radyushkin Chernyak etal

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Second Moment of Pion Distribution Amplitude

\[ < \xi^2 > = \int_{-1}^{1} d\xi \xi^2 \phi(\xi) \]

\[ \xi = 1 - 2x \]

\[ < \xi^2 >_\pi = \frac{1}{5} = 0.20 \]
\[ < \xi^2 >_\pi = \frac{1}{4} = 0.25 \]

\[ \phi_{asympt} \propto x(1 - x) \]
\[ \phi_{AdS/QCD} \propto \sqrt{x(1 - x)} \]

Lattice (I) \[ < \xi^2 >_\pi = 0.28 \pm 0.03 \] Donnellan et al.

Lattice (II) \[ < \xi^2 >_\pi = 0.269 \pm 0.039 \] Braun et al.

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